Geometry & Spatial Sense Module

Ontario Ministry of Education and Training
Queen's Park, Mowat Block,
Toronto, Ontario
M7A 1L2

Impact Math is a professional development program to help teachers of Grades 7/8 implement the new Mathematics curriculum. The program was developed by the Impact Math team at the Ontario Institute for Studies in Education of the University of Toronto (OISE/UT). The development of this resource document was funded by the Ontario Ministry of Education and Training. This document reflects the views of the developers and not necessarily those of the Ministry.

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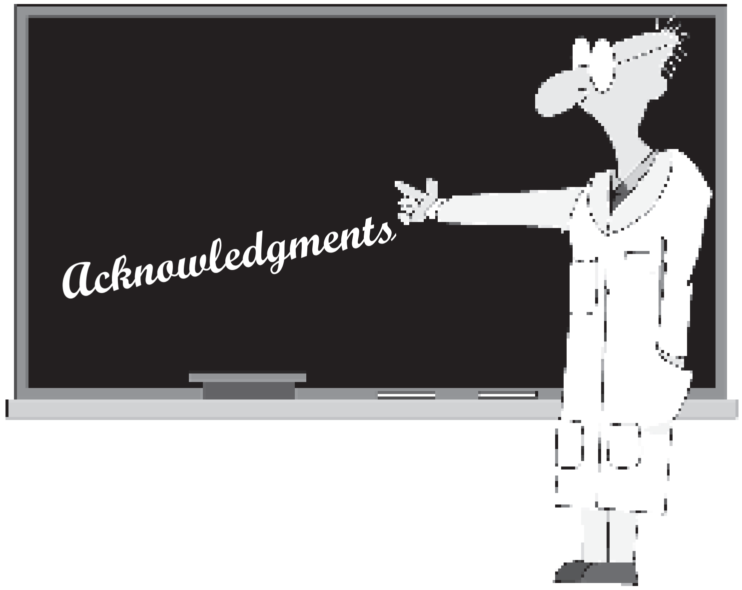

This module is the final product in a series of drafts, revisions and field tests conducted during the 1998–99 school year. Enhancing the usefulness of this booklet is the plethora of wonderful samples of student work that appear under the heading "WHAT YOU MIGHT SEE." For these samples we are deeply indebted to the Grade 7 and 8 students of the following teachers:

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We wish to thank these excellent teachers and their students for their invaluable contribution to this resource. Mike Wierzba and Barry Scully served as the liaisons who conducted the field tests, gathering information from students and teachers and providing insightful suggestions regarding the scoring guides, the activities, and the content of this module.

We acknowledge a significant debt to the following members of the *Impact Math* cadre of trainers who made important suggestions for improvements to the activities and the scoring guides to facilitate their use by teachers and strengthen their use as aids to report card preparation (see pp. 94-95).

Katie Branovacki	Trevor Brown	Jeri Lunney	André Mailloux	Pat Margerm
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A special debt of gratitude is owed to Maria Bodianne, Sandra Dilena, Lorraine Schroetter, and Barry Scully who played a key role in the development of the scoring guides and the selection of student work.

The flow chart on page 12 summarizes a model developed jointly by OISE and the Durham Board of Education for involving students in the creation of their own scoring guides. Permission to use this has been generously granted by Carol Rolheiser who edited the publication that emerged from that collaboration (see p. 12).

The creation of such a document as this involves many stages of revision, rewriting, and reorganization, carrying with it a multitude of opportunities for errors. We are eternally grateful to Rosemary Tanner's sharp eye and expert editorial skills for the many errors and omissions that she purged from the manuscript at each stage.

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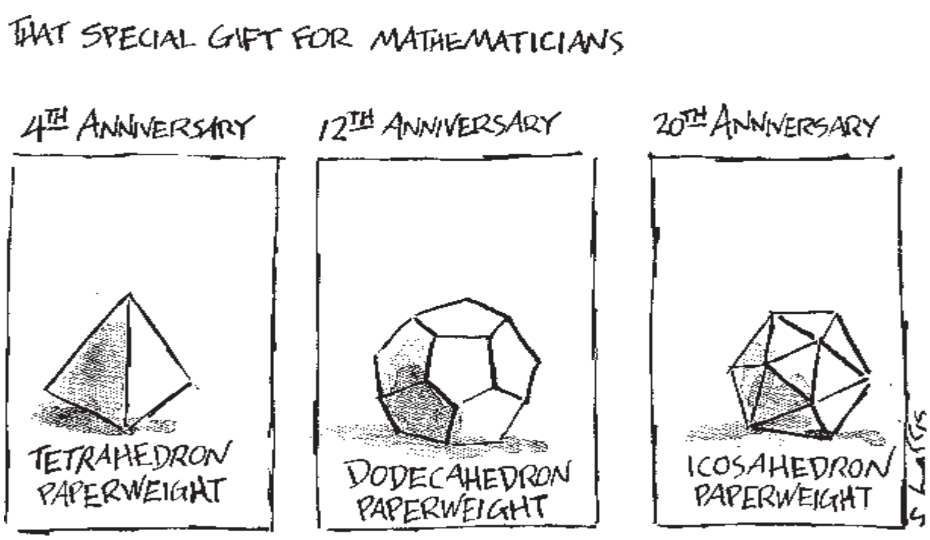

PART III – SPATIAL SENSE IN GRADE 7

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Introduction to the Modules

The Ontario Curriculum, Grades 1– 8: Mathematics, issued in 1997, has redefined the elementary school mathematics curriculum for Ontario. New expectations for student learning require the teaching of new mathematical topics as well as a shift in emphasis of content previously taught. In particular, the new document reflects the growing need for students to expand their skills in processing information, managing data, problem solving, and using technology to achieve these ends. While there is a reduced attention to rehearsing rote skills, such as long division with large divisors or extraction of roots by the formal method, there is a reaffirmation of the need for students to master the multiplication tables and fundamental pencil-and-paper skills that underpin arithmetic facility. Such skills are intended to support the intelligent use of technology in performing complex computations of the type that arise in so-called "real world" contexts.

Implicit in this document is the demand for new or revised methods of instruction and assessment. Educational research of the past twenty years has mounted a compelling argument for a knowledge-building approach to instruction (see page 9) that reduces the role of the teacher as purveyor of information and enhances the teacher's role as facilitator of learning. With this shift in instructional methodology comes a corresponding demand for change in methods of assessment (see pages 10 – 12).

The call for such changes in curriculum, instruction, and assessment has created a need for teachers of Grades seven and eight to plan new programs in mathematics from the plethora of print and electronic resources currently available. Since most of these teachers are responsible for many subject areas in addition to mathematics, the consolidation of these materials into a set of coherent lessons is daunting. To support teachers in this quest, the Ministry of Education and Training has commissioned a set of five modules (of which this is one) that gather together many of the extant resources in a single reference package. Each module addresses one of the five strands in the new curriculum.

Though they address different content strands, all modules have the same format. Part I outlines the rationale underpinning the ideas and activities developed in the module. Part II provides a brief instruction for teachers on the new content or approaches in that strand. Part III provides a set of four sample activities for Grade 7. Together these constitute an authentic task designed to consolidate and extend earlier developmental activities. This unit is intended to model the instructional and assessment philosophies discussed in Part I. **It is not intended to cover the entire content of the strand, nor to replace any resources presently used, but rather to supplement the current program.** Included in Part III under the heading "What You Might See" are samples of student work, classified by achievement level, and presented opposite a rubric that will help you assess the work of your students. Part IV parallels Part III, except it is keyed to the Grade 8 unit. However, it is recommended that all teachers familiarize themselves with the contents of both Parts III and IV. Part IV concludes with a selected list of appropriate print and media resources at the Grade 7 – 8 levels and some useful Internet addresses to fulfill the intent that the module provide a single reference to help teachers implement the new curriculum.

The Rationale For Geometry & Spatial Sense

Spatial Sense

The rationale for the focus on spatial sense in the Ontario mathematics curriculum is given on page 42 of *The Ontario Curriculum, Grades 1– 8: Mathematics:*

> *Spatial sense is the intuitive awareness of one's surroundings and the objects in them. Geometry helps us represent and describe, in an orderly manner, objects and their interrelationships in space. A strong sense of spatial relationships and competence in using the concepts and language of geometry can improve students' understanding of number and measurement.*
>
> *Spatial sense is necessary for interpreting, understanding, and appreciating our inherently geometric world. Insights and intuitions about the characteristics of two-dimensional shapes and three-dimensional figures, the interrelationships of shapes, and the effects of changes to shapes are important aspects of spatial sense.*
>
> *Students need to visualize, draw, and compare shapes in various positions in order to develop their spatial sense.*

The National Council of Teachers of Mathematics (NCTM) provides further elaboration of this conception in *Curriculum and Evaluation Standards for School Mathematics* (p. 112):

> *The study of geometry helps students represent and make sense of the world. Geometric models provide a perspective from which students can analyze and solve problems, and geometric interpretations can help make an abstract (symbolic) representation more easily understood. Many ideas about number and measurement arise from attempts to quantify real-world objects that can be viewed geometrically.*

Geometry

The rationale for the inclusion of some informal coordinate (analytic) geometry (e.g., graphing points in the first quadrant and measurement of distances on a grid) in the Ontario curriculum is also given on page 42 of *The Ontario Curriculum, Grades 1– 8: Mathematics:*

> *…Although students need to learn the formal language of geometry, instruction in the correct terminology should not be the only focus of the program. Students must also explore and understand relationships among figures.*
>
> *… All students should have access to computers and graphing calculators as powerful tools that can help them expand their understanding of analytic geometry.*

The Role of Technology in the Geometry & Spatial Sense Strand

The policy on the use of technology, as embodied in *The Ontario Curriculum, Grades 1– 8: Mathematics,* is stated on page 7 of that document:

> *Students are expected to use calculators or computers to perform operations that are lengthier or more complex than those covered by the pencil-and-paper expectations. When students use calculators and computers to perform operations, they are expected to apply their mental computation and estimation skills in predicting and checking answers. Students will also use calculators and computers in various experimental ways to explore number patterns and to extend problem solving.*

The rationale for this policy is expressed clearly and strongly on page 17 of the *National Council of Teachers of Mathematics 1998–99 Handbook:*

> *Technology has changed the ways in which mathematics is used and has led to the creation of both new and expanded fields of mathematical study. Thus, the technology is driving change in the content of mathematics programs, in methods for mathematics instruction, and in the ways that mathematics is learned and assessed. A vital aspect of such change is a teacher's ability to select and use appropriate instructional technology to develop, enhance, and extend students' understanding and application of mathematics. It is essential that teachers continue to explore the impact of instructional technology and the perspectives it provides on an expanding array of mathematics concepts, skills, and applications.*

The position statement of the NCTM follows this rationale with six recommendations including the following three:

- *Every student should have access to an appropriate calculator.*
- *Every school mathematics program should provide students and teachers access to computers and other appropriate technology for individual, small-group, and whole-class use, as needed, on a daily basis.*
- *Curriculum development, evaluation, and revision must take into account the mathematical opportunities provided by instructional technology. When a curriculum is implemented, time and emphasis must be given to the use of technology to teach mathematics concepts, skills, and applications in the ways they are encountered in an age of ever increasing access to more-powerful technology.*

For the complete position statement including all six recommendations, access the NCTM's web site at www.nctm.org/about or e-mail infocentral@nctm.org.

A vision of how the computer should be used specifically for the teaching and learning of geometry is presented in the NCTM's publication *Curriculum and Evaluation Standards for School Mathematics* (p. 114).

> *Computer software [such as the Geometer's Sketchpad and Cabri Geometry] that allows students to construct geometric figures and determine the measures of arcs, angles, and segments creates a rich environment for the investigation of geometric properties and relationships. Students can make conjectures and explore other figures to verify their reasoning.*

Understanding the Learning Process & Its Impact on Instruction

In this and the other four modules, we present activities that attempt to incorporate a range of instructional approaches. The students are sometimes given information and required to read, interpret, and apply it in an exercise. In other cases, the students must investigate, explore, and discover concepts that lurk beneath the surface of an activity. In some cases, the students will work individually, while in others they will work collaboratively or cooperatively. For example, in Activity 3 in Part III of this module, students work individually to construct the Platonic solids from nets. In the next activity (Activity 4) students are organized in working teams to construct a classroom geodesic clubhouse. This consolidates and extends some of the learning of the previous activities and helps them discover how to work cooperatively in the construction of a tangible product. The activity in Part IV has students searching by trial-and-error methods for the triangle of minimum perimeter inscribed in a given triangle. They are encouraged to work individually or in pairs using geoboards, graph paper, plastic mirrors, and/or dynamical geometry software to discover the optimal solution. Activity 4 requires that each student write a report explaining their findings and their recommendations. This ensures individual accountability while promoting the development of communication skills.

In view of these multiple perspectives on instruction, one might assume that all traditional approaches to teaching will disappear as these philosophies are incorporated. However a response to the question “What should I see in a [NCTM] Standards-based mathematics classroom?” the *NCTM 1997–98 Handbook* presents a balanced and accessible image of effective instruction:

> *First and foremost, you'll see students doing mathematics. But you'll see more than just students completing worksheets. You'll see students interact with one another, use other resources along with textbooks, apply mathematics to real-world problems, and develop strategies to solve complex problems.*
>
> *Teachers still teach. The teacher will pose problems, ask questions that build on students' thinking, and encourage students to explore different solutions. The classroom will have various mathematical and technological tools (such as calculators, computers, and math manipulatives) available for students to use when appropriate. The teacher may move among the students to understand their thinking and how it is reflected in their work, often challenging them to engage in deeper mathematical thinking.*

Assessment: Rubrics & Achievement Levels

The changes in curriculum and instruction described on the preceeding pages have significant implications for assessment and evaluation. Among these implications is the shift from norm-referenced to criterion-referenced assessment, as described on page 1 of *The Assessment Standards for School Mathematics* published by the NCTM in 1995:

> *At present, a new approach to assessment is evolving in many schools and classrooms. Instead of assuming that the purpose of assessment is to rank students on a particular trait, the new approach assumes that high public expectations can be set that every student can strive for and achieve, that different performances can and will meet agreed-on expectations, and that teachers can be fair and consistent judges of diverse student performances.*

The Ontario Curriculum, Grades 1–8: Mathematics (see pp. 4–5) also embraces the move to criterion-referenced assessment and includes four levels of achievement for describing student performance:

> *High achievement is the goal for all students, and teachers, students, and parents need to work together to help students meet the expectations specified. The achievement levels are brief descriptions of four possible levels of student achievement. These descriptions, which are used along with more traditional indicators like letter grades and percentage marks, are among a number of tools that teachers will use to assess students' learning. The achievement levels for mathematics focus on four categories of skills: problem solving, understanding of concepts, application of mathematical procedures, and communication of required knowledge. When teachers use the achievement levels in reporting to parents and speaking with students, they can discuss with them what is required for students to achieve the expectations set for their grade.*

Descriptions of the four levels of achievement for problem solving, concepts, applications, and communication are shown on page 9 of that document. These are the levels for concept understanding:

knowledge/skills	Level 1	Level 2	Level 3	Level 4
Understanding of concepts	**The student shows understanding of concepts:**			
	– with assistance	– independently	– independently	– independently
	– by giving partially complete but inappropriate explanations	– by giving appropriate but incomplete explanations	– by giving both appropriate and complete explanations	– by giving both appropriate and complete explanations and by showing that he or she can apply the concepts in a variety of contexts
	– using only a few of the required concepts	– using more than half the required concepts	– using most of the required concepts	– using all of the required concepts

A table such as the one above that describes levels of achievement is called a *rubric*. Included with the student activities, in this and the other modules, are rubrics and samples of student work that exemplify the levels of student performance as defined in *The Ontario Curriculum, Grades 1–8: Mathematics.*

Assessment: Rubrics & Achievement Levels

The release of the first module in this series, *Data Management & Probability*, was met with widespread enthusiasm. It confirmed our belief that teachers need and want materials to help them implement the new mathematics curriculum. Of particular interest to teachers are the issues associated with assessment and evaluation. The shift in emphasis from rote learning to higher-order processes, such as problem solving, drawing inferences, and communicating mathematical conclusions, requires that methods of performance assessment be added to the battery of devices that teachers use to assess mathematical learning. As observed in the NCTM publication *Curriculum and Evaluation Standards for School Mathematics: Addendum Series – A Core Curriculum* (1992):

> *Questions eliciting open-ended responses require more holistic approaches for scoring. Indirectly, they convey to students the need to communicate their ideas clearly and to construct their responses for a purpose. The impact on the curriculum of this type of assessment is to hold students accountable for demonstrating their understanding of connected ideas rather than displaying their proficiency with disconnected skills.* (p. 11)

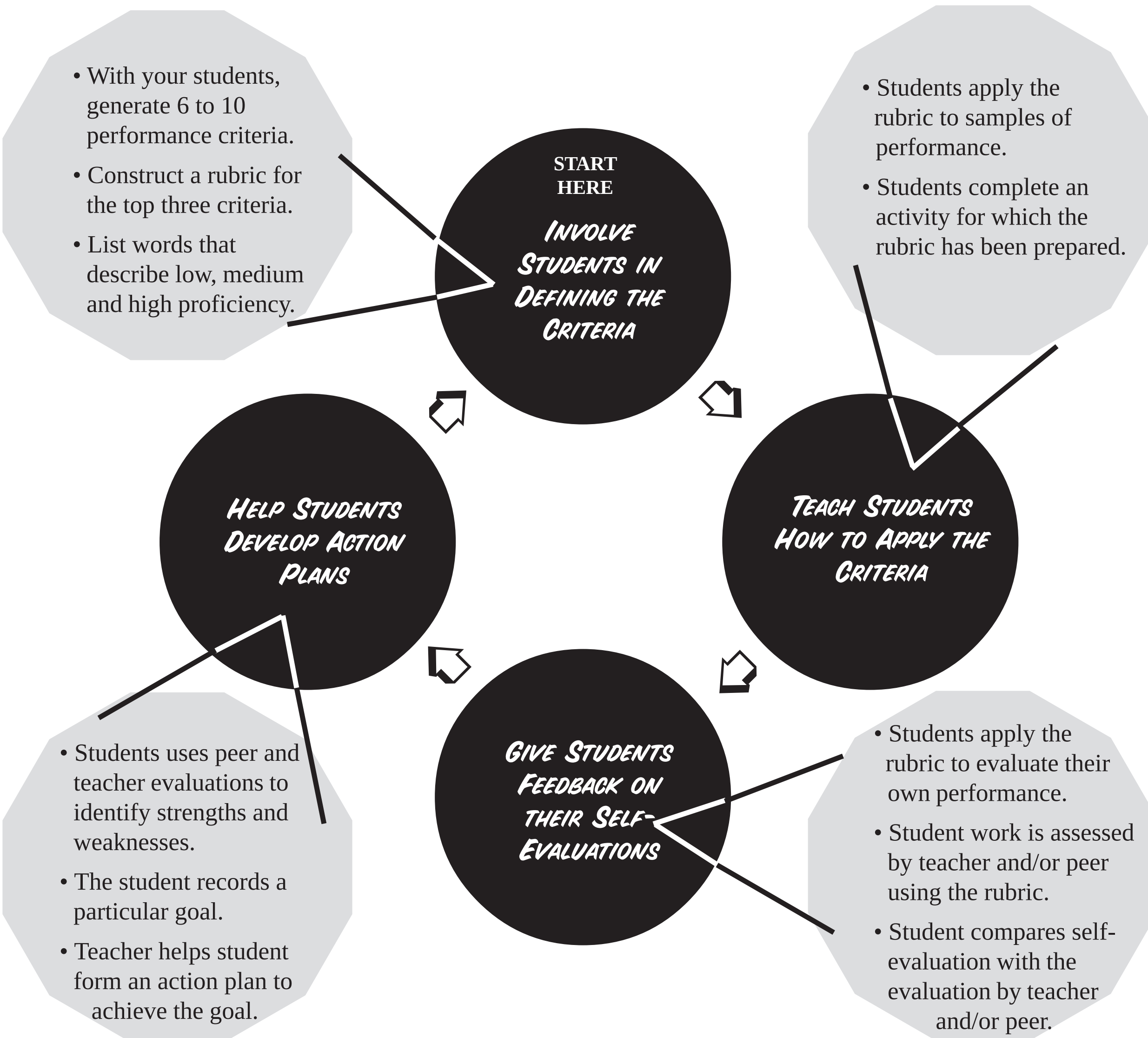
One of the most important devices for the holistic scoring of higher-order tasks is the rubric. The rubric shown on page 10 is an example of what is called a "general rubric." In its publication *Assessment Standards for School Mathematics* (1995), the NCTM defines a general rubric as "an outline for creating task-specific rubrics" (p. 90). Furthermore it defines a "task-specific rubric" as a rubric that "describes levels of performance for a particular complex task and guides the scoring of that task consistent with relevant performance standards." In this module we present, under the heading WHAT YOU MIGHT SEE, samples of student responses to some of the activities. Large samples of student work collected during the field tests of these materials were used to create scoring guides. These guides are task-specific rubrics. You will notice however that they evaluate the "product," i.e., the student work, while the general rubric shown on page 10 includes an observational component of assessment (e.g., "with assistance," "independently"). Since there can be no observational component in the assessment of *completed* student work, the scoring guides in this book do not use phrases such as "independently" or "with assistance." **It is expected that teachers will use each scoring guide as a starting point in the development of a task-specific rubric that will evolve as it is used with students.**

On page 12, we offer some suggestions on how to develop task-specific scoring guides. However, it is important to recognize that the creation of rubrics is highly subjective and is more an art than a science. In the *TIMSS Monograph #1: Curriculum Frameworks for Mathematics and Science* (1993), Robitaille et al. issue this caveat:

> *Measuring educational achievement is difficult from both a conceptual and a practical perspective. What counts as "achievement" is not always easy to discern and even when a concept of achievement has been clearly explicated, ways and means for assessing it are not easily devised. The ongoing debate about educational measurement and the increasing number of alternative assessment approaches proposed in educational circles attest to this problem.* (p. 36)

Involving Students in Creating Their Own Scoring Guides:
a four-step model

The *Patterns and Algebra* module (p. 12) presented a model for creating scoring guides. Teachers have found that student performance is often enhanced when this process is taken one step further and students participate in the construction of their own scoring guides. The four-step model presented below summarizes a procedure presented in *Self-Evaluation–Helping Students Get Better at It!* (cited below). This excellent resource is a joint publication of OISE/UT and the Durham Board of Education.



Helpful Resources for Creating Scoring Guides

Bryant, Deborah and Mark Driscoll. *Exploring Classroom Assessment in Mathematics.* Reston, VA: National Council of Teachers of Mathematics, 1998.

Danielson, Charlotte. *A Collection of Performance Tasks and Rubrics: Upper Elementary School Mathematics.* Larchmont, NY: Eye on Education, Inc., 1997.

Flewelling, Gary and Chuck Lemenchik. *Mathematics Assessment: Grades 7 & 8.* Toronto, ON: Gage Educational Publishing Company, 1997.

Rolheiser, Carol (Ed.) *Self-Evaluation–Helping Students Get Better at It!* Toronto, ON: VisuTronX, 1996.

PART II

What's New in Geometry & Spatial Sense?

A Hands-On Approach to Geometry

New instructional approaches rather than new content represent the major change in the geometry and spatial sense strand. The NCTM in its important publication *Curriculum and Evaluation Standards for School Mathematics* identified thirteen curriculum "standards" for grades 5–8 mathematics. Standard 12, titled "geometry," presents a rationale for a "hands-on" approach to developing spatial sense and geometric insights through exploring, conjecturing, and drawing inferences (p. 112):

> *Students discover relationships and develop spatial sense by constructing, drawing, measuring, visualizing, comparing, transforming, and classifying geometric figures. Discussing ideas, conjecturing and testing hypotheses precede the development of more formal summary statements. In the process, definitions become meaningful, relationships among figures are understood, and students are prepared to use these ideas to develop informal arguments.*

The NCTM provided further elaboration of this conception in *Curriculum and Evaluation Standards for School Mathematics: Addenda Series – Geometry in the Middle Grades* (p. 3):

> *Problems that allow for student explorations with materials and manipulatives are especially appealing to middle grade students. To develop concepts and to encourage creative explorations, it is essential that students use tangrams, geoboards, centicubes, Miras [semi-transparent plastic mirrors] grids… and two- and three-dimensional models.*

The Ontario Curriculum, Grades 1–8: Mathematics, includes in its Grade 7 and 8 expectations in Geometry and Spatial Sense the requirement that students construct three-dimensional models of geometric figures. Such investigations naturally lead to the discovery of a beautiful and important property of solids known as Euler's Theorem. Although this theorem may be known by many teachers, we include it here because it appears often in the Grade 7 sample unit in Part III.

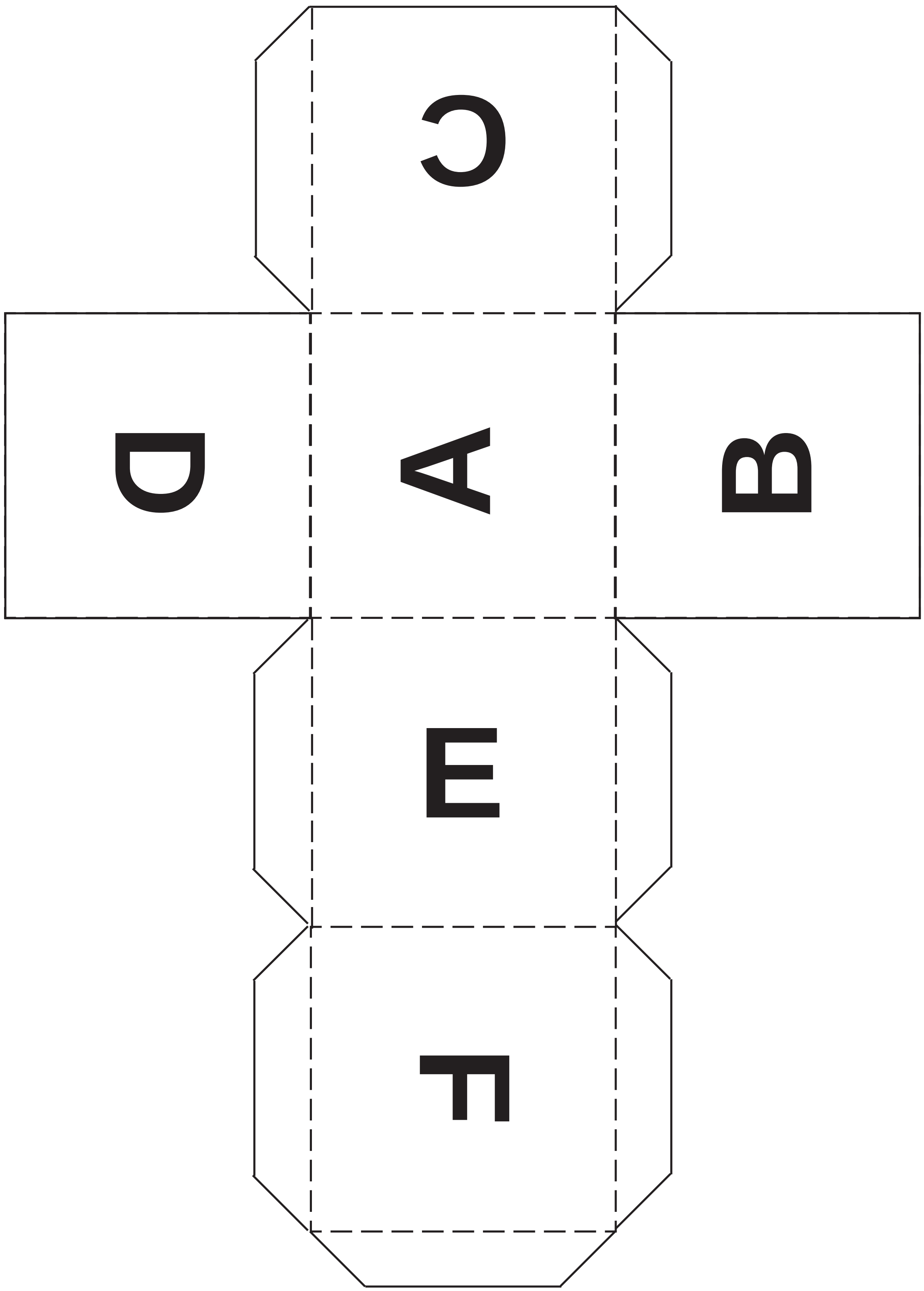
Euler's Theorem *If E, F, and V denote respectively the number of edges, faces, and vertices of a polyhedron, then*

$$F + V = E + 2$$

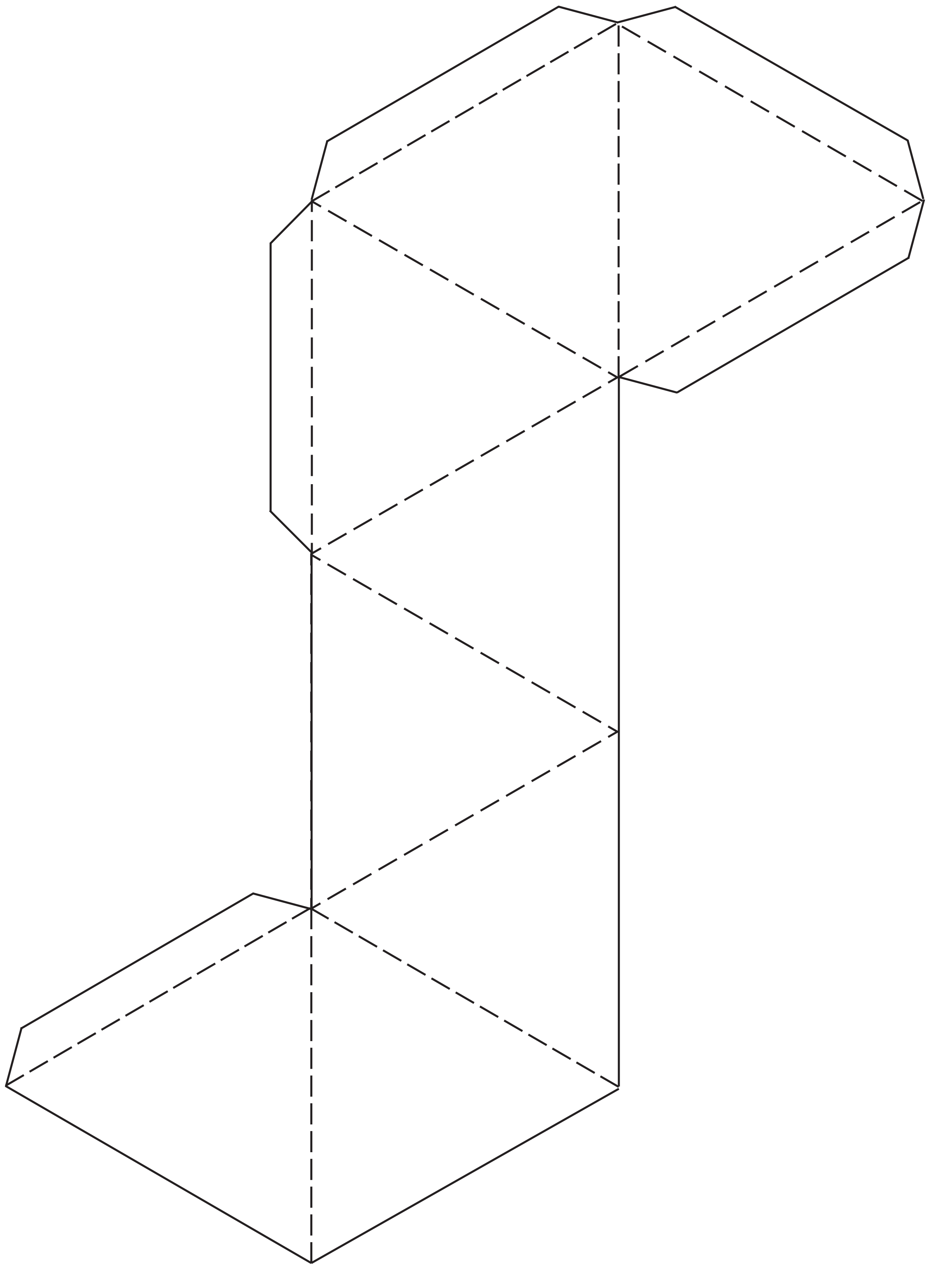
Euler's Theorem states that the number of faces plus the number of vertices of any polyhedron (without a hole in it) is two more than its number of edges. In the Grade 7 sample activities, students construct several polyhedra from their nets and discover Euler's theorem. Then they construct a geodesic club house as a class project and check whether it satisfies Euler's Theorem. In the process, they visualize three-dimensional figures from various perspectives and conjecture the kinds of shadows they might cast.

In the Grade 8 sample unit, students use geoboards, trial-and-error, and formal computation to find the triangle of minimum perimeter that can be inscribed in a given triangle. The problem is embedded in the context of finding a minimal network for a subway system, and it ultimately involves students in performing reflections on a grid and applying angle relationships from elementary geometry. Three parallel investigations using *The Geometer's Sketchpad* are included for classrooms equipped with the appropriate software.

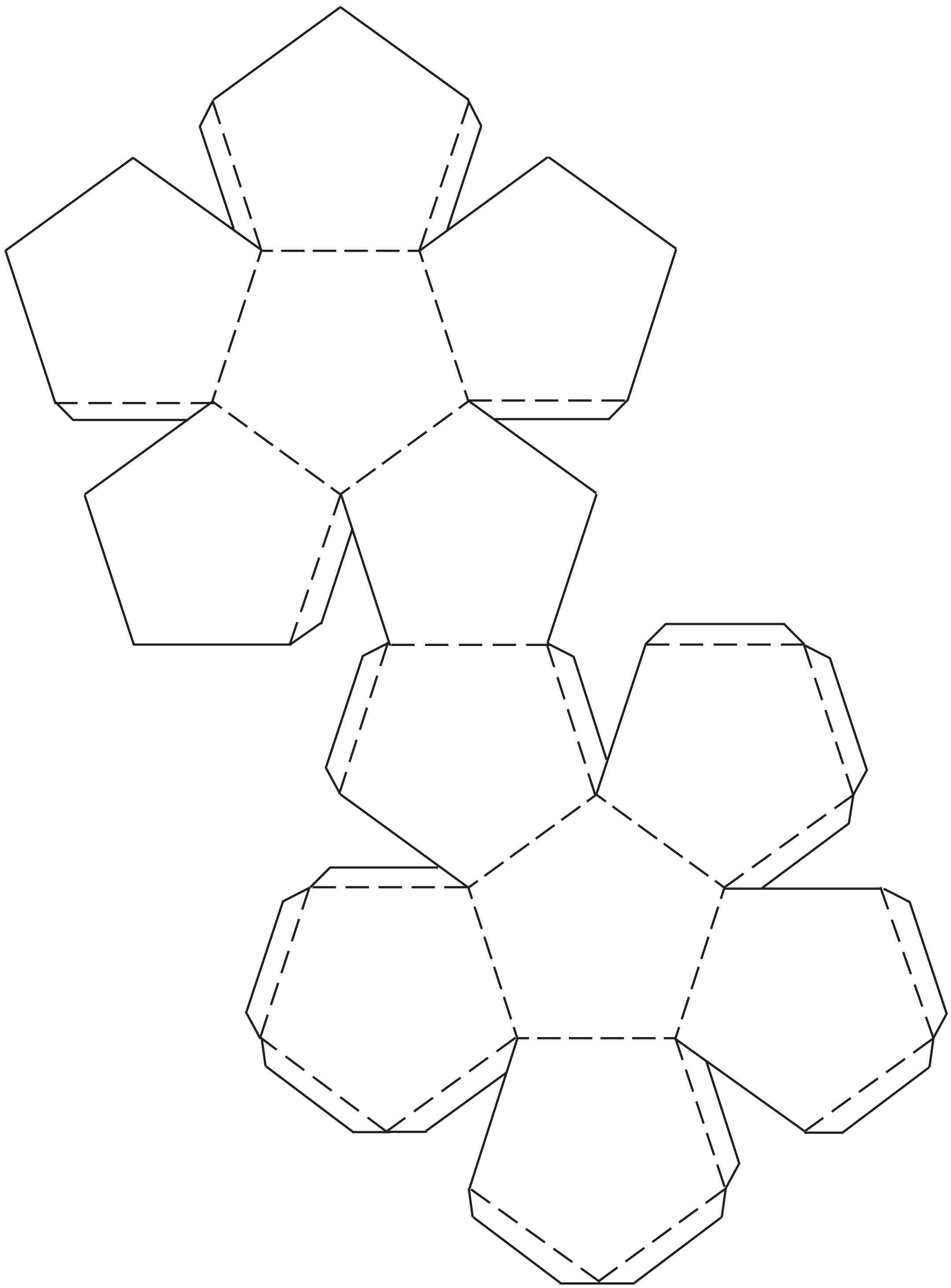
A Net for a Lettered Cube



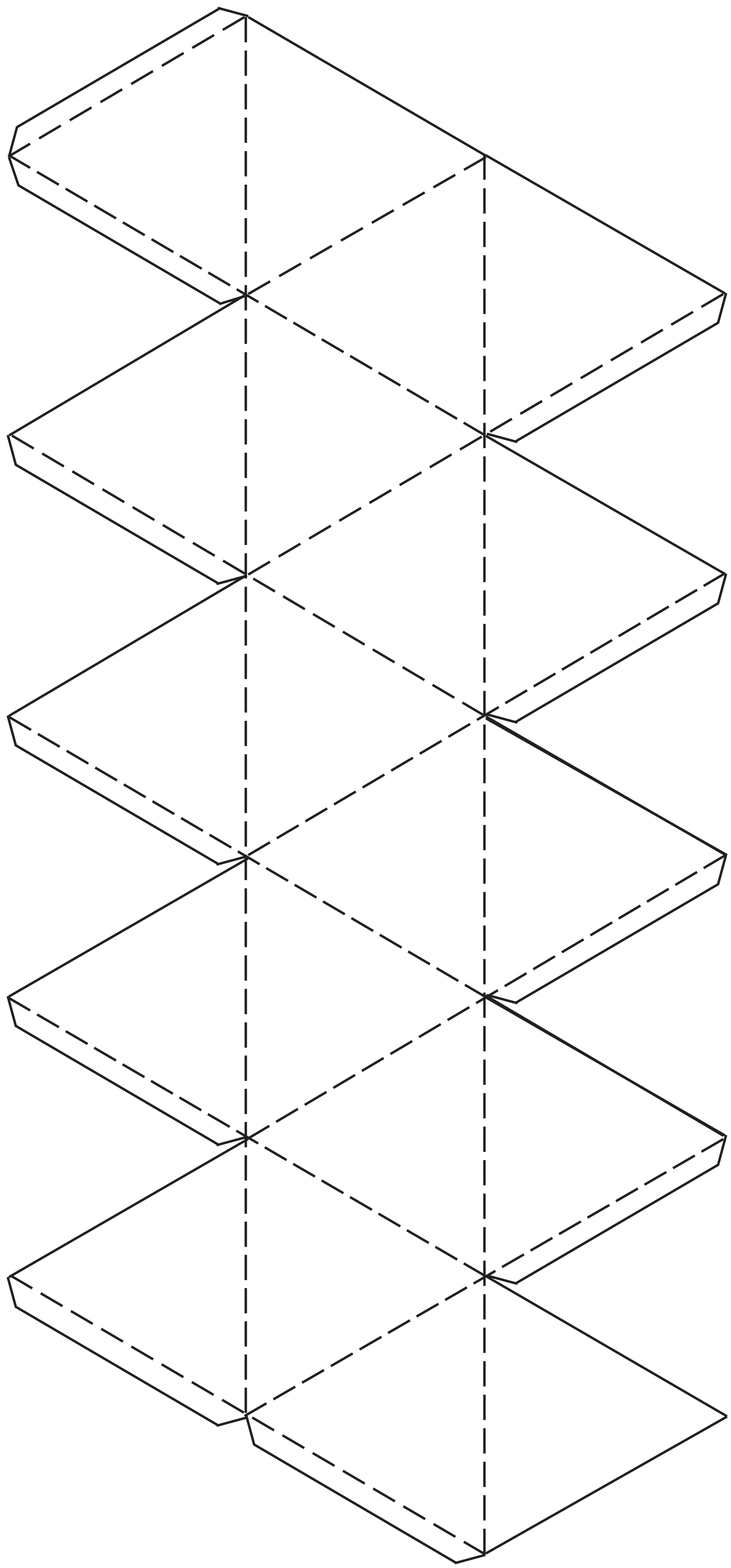
A Net for an Octahedron



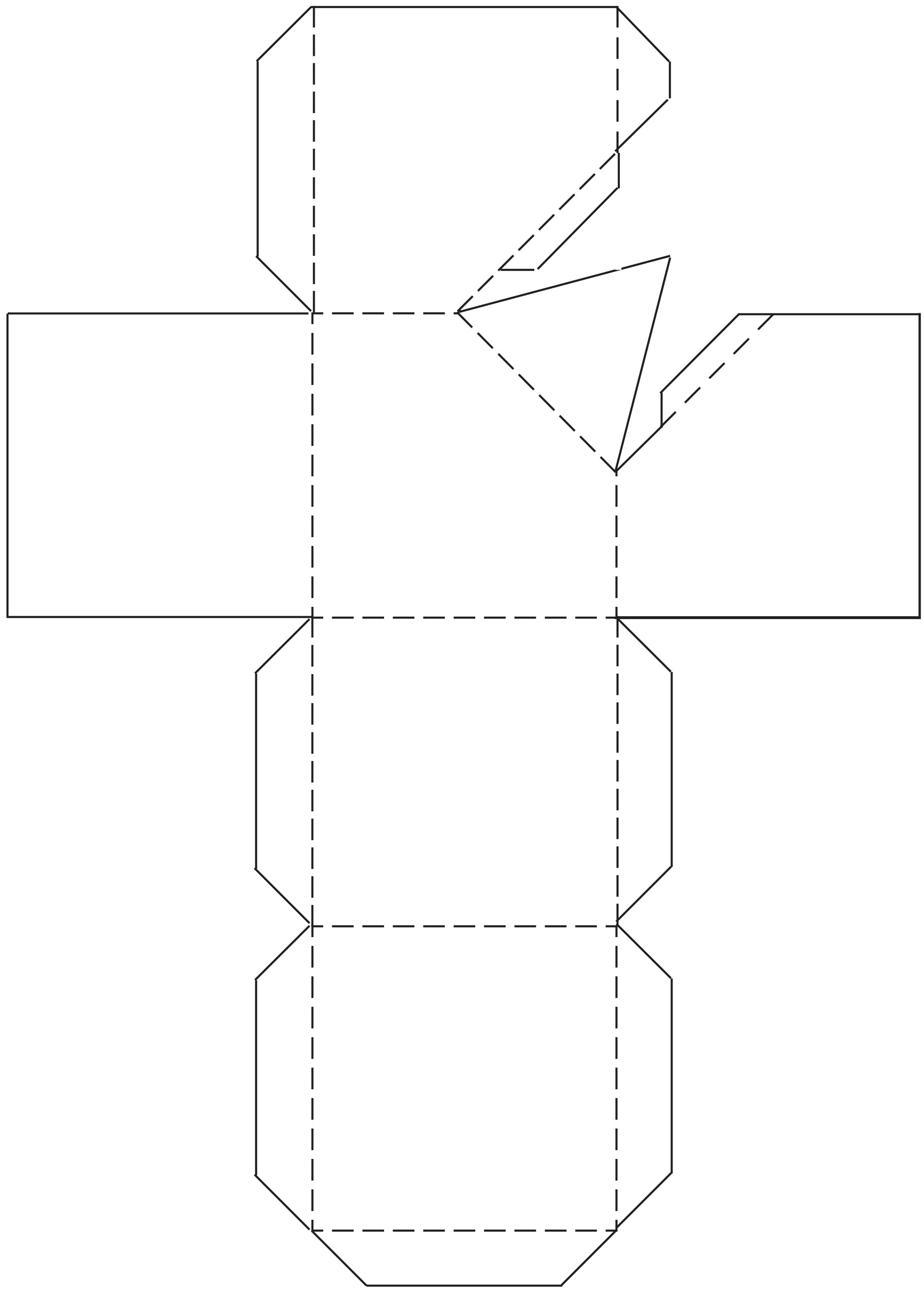
A Net for a Dodecahedron



A Net for an Icosahedron

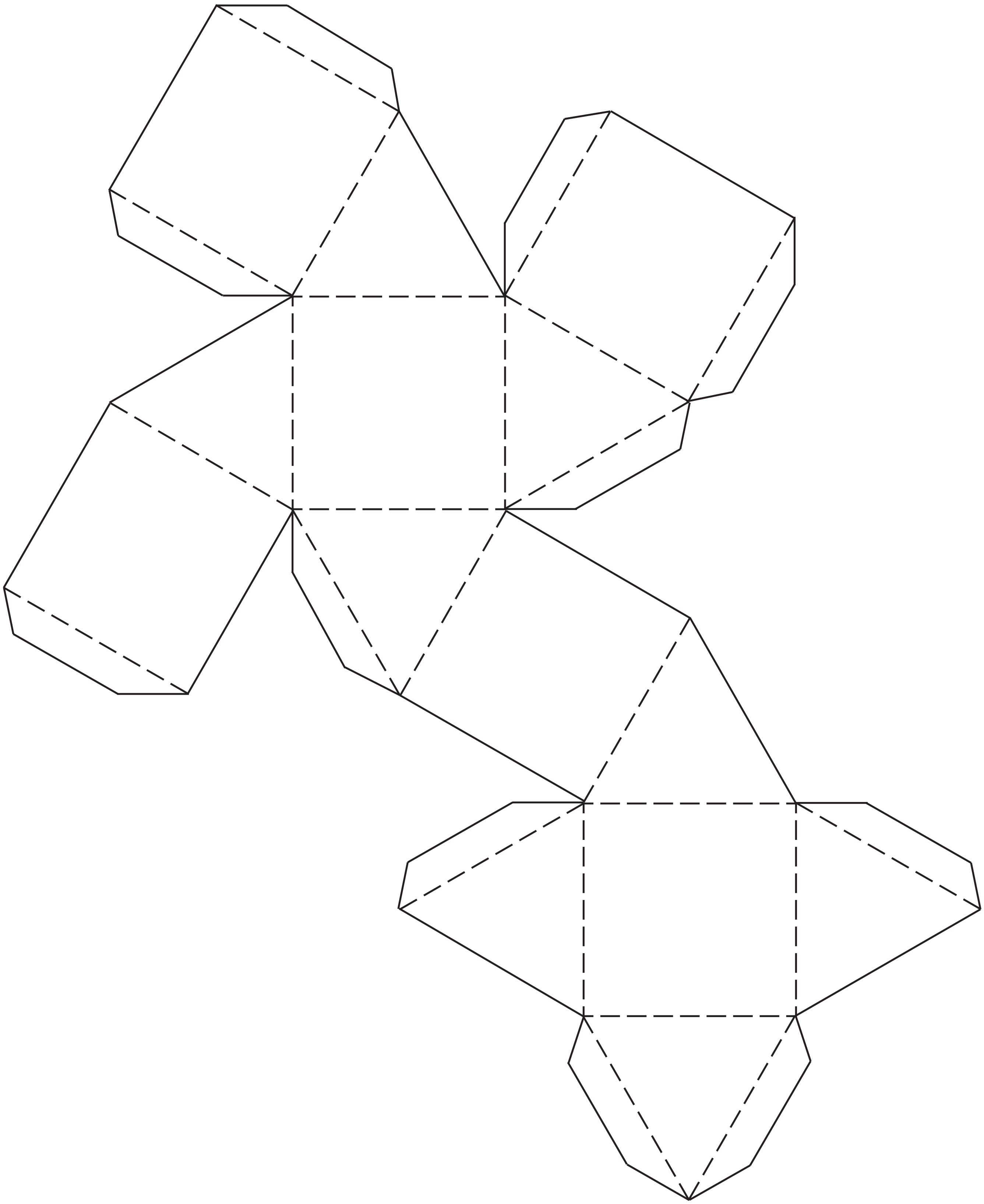


A Net for a Truncated Cube



TEMPLATE

A Net for a Cuboctahedron



PART III

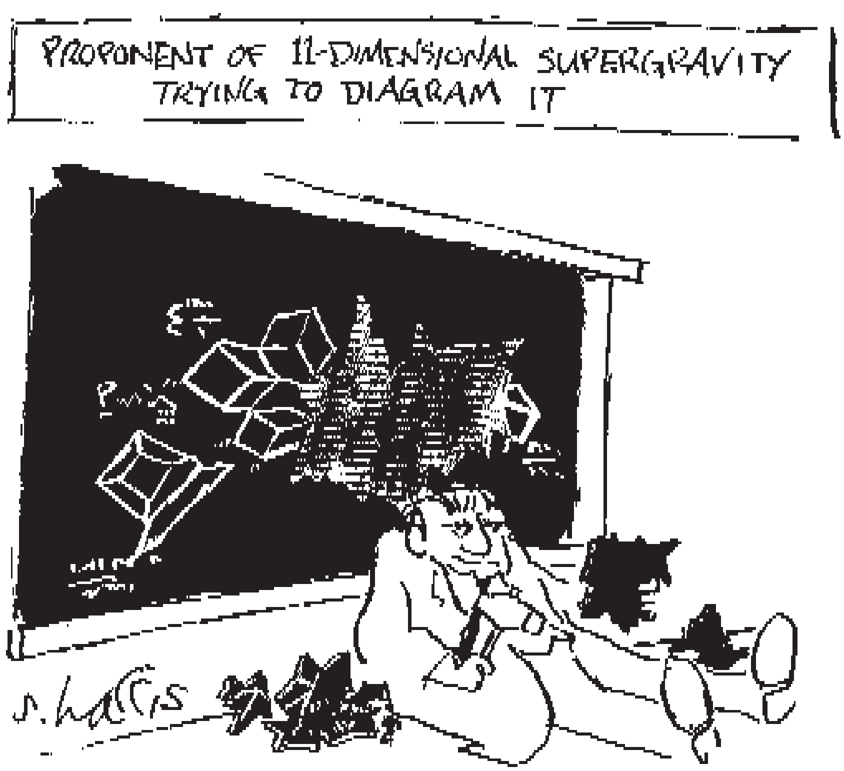
Spatial Sense in Grade 7

Geometry & Spatial Sense: Grade 7

Overall Expectations

By the end of Grade 7, students will:

- identify, describe, compare, and classify geometric figures;
- identify, draw, and construct three-dimensional geometric figures from nets;
- identify congruent and similar figures;
- explore transformations of geometric figures;
- understand, apply, and analyse key concepts in transformational geometry using concrete materials and drawings;
- use mathematical language effectively to describe geometric concepts, reasoning, and investigations.



Specific Expectations

(For convenient reference, the specific expectations are coded. G 7-1 refers to the first Geometry and Spatial Sense expectation in Grade 7.)

Students will:

Three- and Two-Dimensional Geometry

G 7-1 - recognize the front, side, and back views of three-dimensional figures;

G 7-2 - sketch front, top, and side views of three-dimensional figures with or without the use of a computer application;

G 7-3 - sketch three-dimensional objects from models and drawings;

G 7-4 - build three-dimensional figures and objects from nets;

G 7-5 - identify two-dimensional shapes that meet certain criteria (e.g., an isosceles triangle with a 40˚ angle);

G 7-6 - explain why two shapes are congruent;

G 7-7 - identify through investigation the conditions that make two shapes congruent;

G 7-8 - create and solve problems involving the congruence of shapes;

Transformational Geometry

G 7-9 - recognize the image of a two-dimensional shape under a translation, a reflection, and a rotation in a variety of contexts;

G 7-10 - create and analyse designs that include translated, rotated, and reflected two-dimensional images using concrete materials and drawings, and using appropriate computer applications;

G 7-11 - identify whether a figure will tile a plane;

G 7-12 - construct and analyse tiling patterns with congruent tiles;

G 7-13 - describe designs in terms of images that are congruent, translated, rotated, and reflected.

Activity 1 – Teacher Edition

How Many Faces Has Cinesphere?

Expectations Addressed

G 7-1 recognize the front, side, and back views of three-dimensional figures.

G 7-2 sketch front, top, and side views of three-dimensional figures with or without the use of a computer application.

G 7-6 explain why two shapes are congruent.

G 7-7 identify through investigation the conditions that make two shapes congruent.

G 7-8 create and solve problems involving the congruence of shapes.

Context

The context for this unit has been chosen to support the Structures and Mechanism strand of the Grade 7 program defined in *The Ontario Curriculum, Grades 1–8: Science and Technology.* The essence of that strand is defined on page 84 of that document:

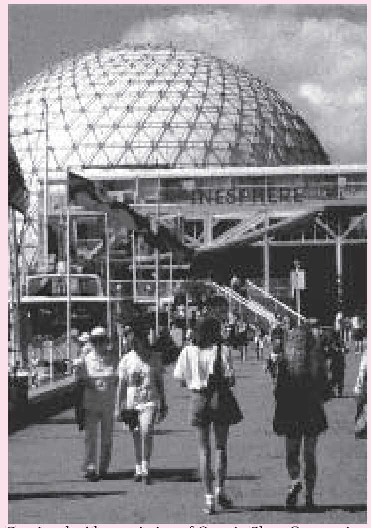
> *Using increasingly sophisticated techniques, students will continue to investigate how different structural forms support or withstand loads by designing, building and testing solid structures, shell structures, and frame structures.*

Before embarking upon this unit, ensure that students have mastered the expectations for Grade 6 Geometry as delineated on pages 48 and 49 of *The Ontario Curriculum, Grades 1–8: Mathematics.* In particular, students can prepare for this unit by building, visualizing and sketching 3-D figures with interlocking cubes. (See *Spatial Visualization Warm-Up* Template, p. 54.)

Activity 1 opens with a review of the concept of congruence. Students are challenged to divide a 3×3 grid into two congruent halves in as many ways as possible. To provide a transition from two-dimensional shapes to three-dimensional figures, students are guided to visualize a truncated cube and cuboctahedron and to enumerate their edges, faces, and vertices. Activity 1 culminates in the students' construction of the truncated cube and the cuboctahedron from provided templates (see pp. 19, 20). This enables them to check the answers they obtained by visualiztion.

Activity 1 – Student Page

How Many Faces Has Cinesphere?



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In 1954, American inventor Buckminster Fuller patented the *geodesic dome*. A geodesic dome is a structure that is composed of triangles joined together to form all or part of a spherical shell. Fuller knew that a sphere is the most efficient shape for containing space. He sought to design a structure that would be strong and yet economical.

The photograph shows one of the first public geodesic domes, Cinesphere, located at Ontario Place in Toronto. Cinesphere is composed of metal bars called *struts* that form triangular *faces* covering a spherical surface of diameter 38 m. (A *face* is any triangle formed from 3 struts and the point where two or more struts are joined is called a *vertex*.) When it opened in 1971, Cinesphere was the first permanent IMAX film theatre in the world. Its large screen is approximately 19 m wide and 24 m high and it seats 752 people. Since the creation of Cinesphere, geodesic domes for large-screen films are springing up everywhere. One of the most famous examples is the Spaceship Earth pavilion at the Disney Epcot Center. It has a diameter of 80 m!

In Activity 2, students construct frame structures of pyramids and prisms of various types and are guided to the discovery of Euler's Theorem (see p. 14) for these figures. Activity 3 provides templates for the construction of each of the five Platonic solids. By counting the edges, faces, and vertices of their models, students extend Euler's theorem to these figures. Then in Activity 4, they construct their own classroom geodesic dome and test Euler's Theorem for this structure. Finally, the students investigate the properties of "icosahedral" geodesic domes and determine whether their classroom geodesic dome is one of these.

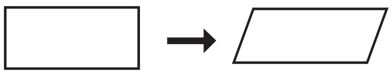
Activity 1 – Teacher Edition

The Lesson Launch

5 minutes

Ask for a volunteer to read aloud the brief historical note on page 26. Ask students what polygons they can see on the surface of Cinesphere. When the triangle is identified, ask why this appears in the construction of so many structures including buildings, bridges, and towers. Develop the idea that the triangle is a strong shape because (unlike the parallelogram) its shape cannot be changed without bending one of its sides. Revisit this concept when teaching the Structural Strength and Stability unit in your Grade 7 science program.

A parallelogram can be deformed without changing the lengths of its sides.



Initiating Activity

20 minutes

Divide students into groups of four. Distribute four copies of page 27 and four pieces of squared paper (see template p. 56) to each group. Before assigning Exercise ❶ on page 27 to the groups, ensure that they understand what is meant by “dividing the grid into two congruent polygons.” Stress that each side of the polygon must join two grid points and that the shaded portion, if cut out, should coincide with the unshaded portion when laid on top of it. Students in each group might work independently and after about 15 minutes, share their work to eliminate redundant cases. When all groups have finished, have each group display their solutions on the overhead or blackboard so that others can add to their own discoveries. Show some solutions (see the 13 solutions, p. 28) that the students may have overlooked.

Ask students whether they think all the triangular faces of Cinesphere are congruent and why (or why not). Entertain a discussion until the idea emerges that two triangles are congruent if (and only if) they have identical shape and size. (See the answer to ❷ b) on p. 28.) Following the discussion, have the students complete Exercise ❷ in their notebooks.

Paired Activity

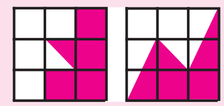
15 minutes

Divide students into pairs. Explain what is meant by a “view from the top,” “a view from the side,” and “a view from the edge.” Then assign Exercises ❸ and ❹. Encourage students to try to visualize these figures, but warn them not to become frustrated because this is a very challenging activity.

Activity 1 – Student Page

How Many Faces Has Cinesphere?

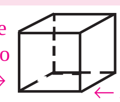
❶ a) Describe what it means when we say that two polygons are congruent.
b) How can you test whether two polygons are congruent?
c) Suppose you and your friend plan to construct identical geodesic domes from triangles. What is the fewest number of measurements of your triangle you would need to communicate to your friend by telephone to ensure that both your triangles were congruent? Explain.
d) How many ways can you divide a 3×3 grid into two congruent polygons **with all vertices as grid points**? Two examples are shown here.



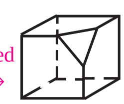
❷ Study the picture of Cinesphere.
a) Name the different kinds of polygons that are formed by the struts.
b) Do you think all the triangular faces of Cinesphere are congruent? Explain why or why not.
c) Do you think the faces on Cinesphere are equilateral triangles? Explain why or why not.
d) Estimate the number of faces on Cinesphere. Describe how you obtained your estimate. Use this number to estimate how many struts on the surface of Cinesphere. Explain how you obtained your estimate.
e) Estimate the number of vertices on the surface of Cinesphere. Explain how you obtained your estimate.

Engineers and architects often use three-dimensional visualization to sketch plans for elaborate structures such as Cinesphere. Try your visualization skills in the following activities.

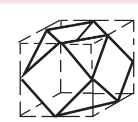
❸ A cube has 6 faces, 8 vertices, and 12 edges. The cube is then *truncated* by slicing off one of its vertices. The newly formed triangular face joins the midpoints of three edges of the original cube.



a) Is the triangular face equilateral? Explain why or why not.
b) How many faces, vertices and edges has the truncated cube?
c) Sketch an outline of the truncated cube as it would appear from each of the three lines of sight shown.



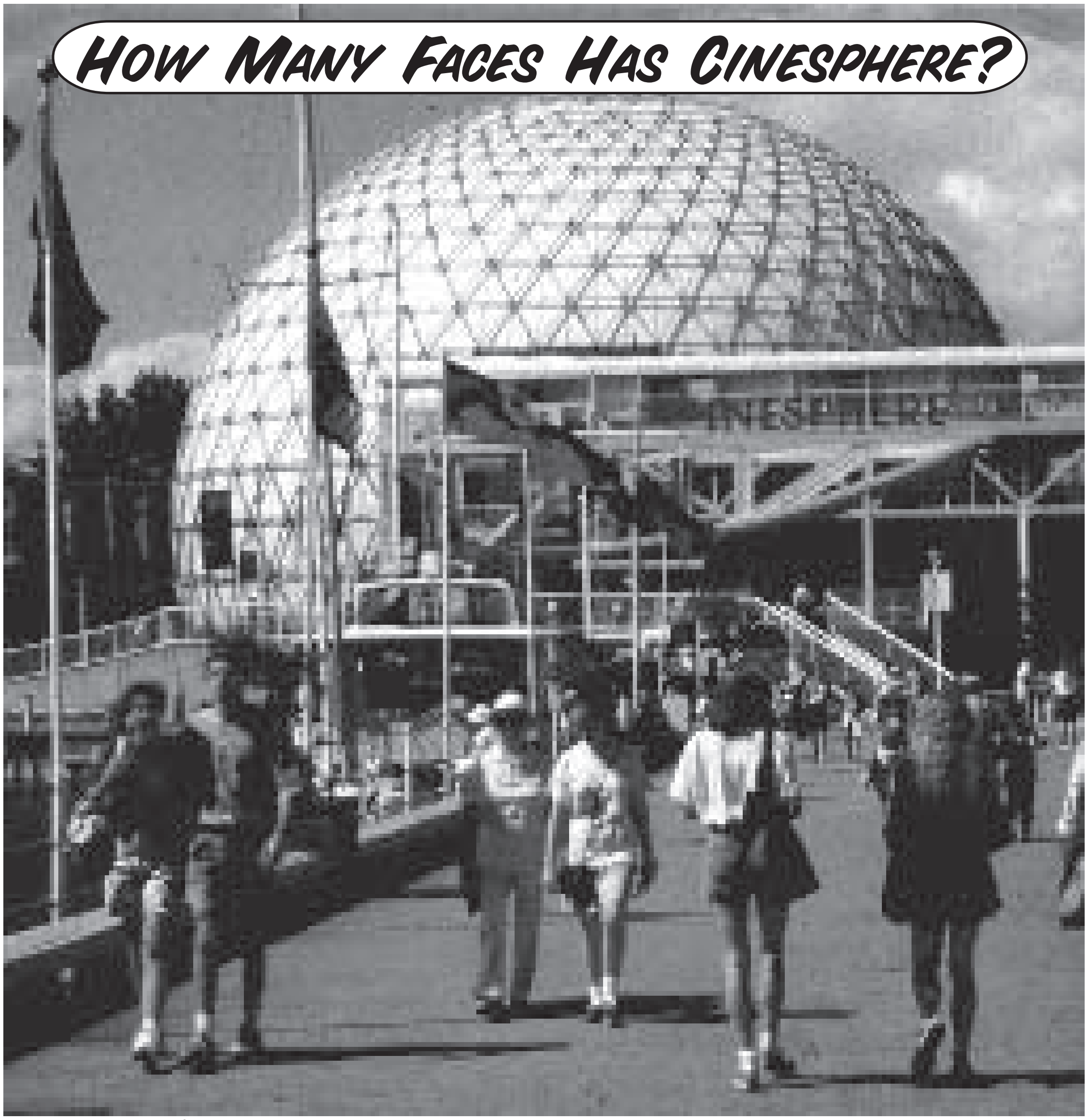
❹ Suppose all 8 of the vertices of a cube are truncated as in Exercise ❸. The newly formed solid is called a *cuboctahedron.*
a) How many faces, vertices, and edges has the cuboctahedron?
b) Sketch an outline of the cuboctahedron as it would appear from each of the three lines of sight shown in Exercise ❸. (See p. 27).


A cuboctahedron

Closure

Initiate a class discussion, inviting students to explain how they counted edges, faces, and vertices of a three-dimensional figure they couldn’t see. Ask students how they might check whether the numbers they obtained were correct. Distribute the templates for the truncated cube and cuboctahedron (see pp. 19, 20). Invite the students, working in their pairs, to build a model of the truncated cube to check their answers to Exercise ❸. Advise them to cut along solid lines on the template and fold along the dotted lines. Ask whether they think there is a relationship between the number of faces, vertices, and edges. Encourage students who wish to construct the cuboctahedron to verify their answers to Exercise ❹.

How Many Faces Has Cinesphere?



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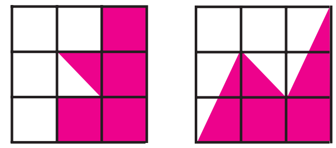
In 1954, American inventor Buckminster Fuller patented the *geodesic dome*. A geodesic dome is a structure that is composed of triangles joined together to form all or part of a spherical shell. Fuller knew that a sphere is the most efficient shape for containing space. He sought to design a structure that would be strong and yet economical.

The photograph shows one of the first public geodesic domes, Cinesphere, located at Ontario Place in Toronto. Cinesphere is composed of metal bars called *struts* that form triangular *faces* covering a spherical surface of diameter 38 m. (A *face* is any triangle formed from 3 struts and the point where two or more struts are joined is called a *vertex*.) When it opened in 1971, Cinesphere was the first permanent IMAX film theatre in the world. Its large screen is approximately 19 m wide and 24 m high and it seats 752 people. Since the creation of Cinesphere, geodesic domes for large screen films are springing up everywhere. One of the most famous examples is the Spaceship Earth pavilion at the Disney Epcot Center. It has a diameter of 80 m!

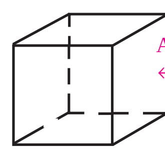
❶ a) Describe what it means when we say that two polygons are congruent.
b) How can you test whether two polygons are congruent?
c) Suppose you and your friend plan to construct identical geodesic domes from triangles. What is the fewest number of measurements of your triangle that you would need to communicate to your friend by telephone to ensure that both your triangles were congruent? Explain.
d) How many ways can you divide a 3×3 grid into two congruent polygons **with all vertices as grid points**? Two examples are shown here.

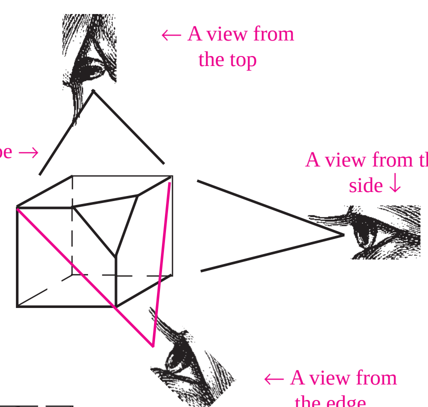


❷ Study the picture of Cinesphere.
a) Name the different kinds of polygons that are formed by the struts.
b) Do you think all the triangular faces of Cinesphere are congruent? Explain why or why not.
c) Do you think the faces on Cinesphere are equilateral triangles? Explain why or why not.
d) Estimate the number of faces on Cinesphere. Describe how you obtained your estimate. Use this number to estimate how many struts on the surface of Cinesphere.
Explain how you obtained your estimate.
e) Estimate the number of vertices on the surface of Cinesphere. Explain how you obtained your estimate.

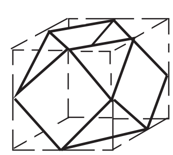
Engineers and architects often use three-dimensional visualization to sketch plans for elaborate structures such as Cinesphere. Try your visualization skills in the following activities.

❸ A cube has 6 faces, 8 vertices, and 12 edges. The cube is then *truncated* by slicing off one of its vertices. The newly formed triangular face joins the midpoints of three edges of the original cube.
a) Is the triangular face equilateral? Explain why or why not.
b) How many faces, vertices, and edges has the truncated cube?
c) Sketch an outline of the truncated cube as it would appear from each of the three lines of sight shown.





❹ Suppose all 8 of the vertices of a cube are truncated as in Exercise ❸. The newly formed solid is called a *cuboctahedron.*
a) How many faces, vertices and edges has the cuboctahedron?
b) Sketch an outline of the cuboctahedron as it would appear from each of the three lines of sight shown in Exercise ❸.


A cuboctahedron

Note:
The dotted lines are edges that would not be visible

Grade 7
Answer Key for Activity 1

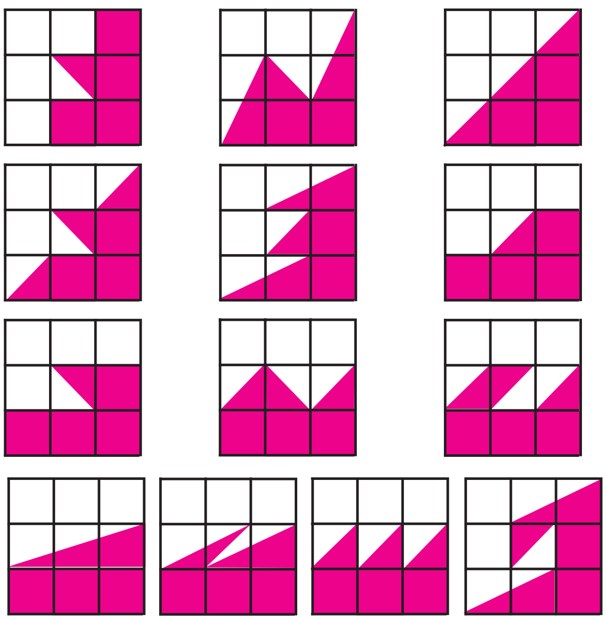
❶ a) Two polygons are congruent if they have identical size and shape.

b) Two polygons are congruent if and only if they coincide when one is placed on top of the other.

c) Any one of these three pieces of information are sufficient to define a congruent triangle:

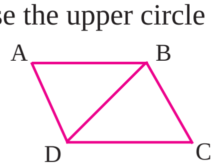
- the lengths of the 3 sides.
- the lengths of 2 sides and the contained angle.
- the measures of 2 angles and the length of a designated side.

d) There are 13 different ways, not counting rotations or reflections of those shown below.



❷ a) Polygons of virtually any number of sides can be formed by joining the vertices on the surface of the sphere. Students should identify at least three different kinds, such as triangle, quadrilateral, and hexagon.

b) Most students will observe that the triangles near the top of the sphere appear smaller than those near the equator, and therefore conclude that the triangles are not all congruent. A more rigorous argument (but beyond most students at this level) is that a quadrilateral between two "circles of latitude" cannot be a parallelogram because the upper circle of latitude has smaller circumference and so AB < CD. Therefore ΔABD is not congruent to ΔCDB.



❷ c) If all the faces were equilateral triangles, then any pair of triangles sharing a common side would be congruent. But we observed in ❷ b) that ΔABD is not congruent to ΔCDB. Therefore, the faces on Cinesphere cannot be all equilateral triangles.

d) Any answer between 800 and 2400 is reasonable depending upon the assumptions made. (There are about 15 "rows" of 90 triangles above the water, accounting for 1350 triangles visible.) Do we count triangular faces that would be present if the doorway were not there? Are there triangles below the water level? The intent of this exercise is to have students identify some assumptions and then demonstrate an organized approach to estimating the number of triangular faces.

There are 3 struts for each triangular face, but each strut bounds 2 faces. So to obtain the number of struts, we multiply the number of faces by 3 and divide the product by 2, to avoid counting each strut twice. We are essentially multiplying the number of faces by 3/2.

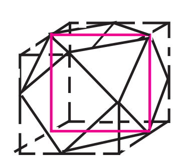
e) There are 3 vertices for each triangular face, but each face is shared by 6 triangles. Therefore to obtain the number of vertices, we multiply the number of faces by 3 and divide the product by 6. This is the same as merely dividing the estimated number of faces by 2.

❸ a) Each side of the triangular face joins the midpoints of two intersecting edges of the cube. Since the midpoints of intersecting edges of the cube are all the same distance apart (by symmetry), then all the sides of the triangular face are the same length. Therefore the triangular face is equilateral.

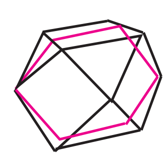
b) Truncated cube: 7 faces, 10 vertices, and 15 edges.

c) The cube will appear to be a square when viewed from the top or side, but a rectangle from the edge.

❹ a) The cuboctahedron has 14 faces, 12 vertices, and 24 edges.



b) The coloured line shows the intersection of a plane perpendicular to the line of sight, showing that the cuboctahedron appears to be a square from the top or side and a hexagon from the edge. (Observe that the hexagon inside the cuboctahedron has all sides of equal length, but a silhouette or shadow projection would be an irregular hexagon.) However, it is sufficient that students recognize that the edge view is a hexagon.



The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 1

	Level 1	Level 2	Level 3	Level 4
PROBLEM SOLVING **Estimation by Organized Counting** (exercise ❷ d & e) N 7-16	•The estimate of the number of faces of Cinesphere is unreasonable. •There is no evidence of a counting strategy for the number of faces.	•The estimate of the number of faces of Cinesphere is reasonable (800–2400). •An appropriate strategy for estimating the number of faces is given.	•The estimate of the number of faces of Cinesphere is reasonable (800–2400). •An appropriate strategy for estimating the number of faces is given, and a correct strategy for estimating the number of struts **or** vertices is evident.	•The estimate of the number of faces of Cinesphere is reasonable (800–2400). •An appropriate strategy for estimating the number of faces is given, and a correct strategy for estimating the number of struts **and** vertices is evident.
CONCEPTS **Understanding of the Concept of Congruence of 2-D Shapes.** (exercises ❶, ❷ b, c) G 7-6, 7-7, & 7-8	•Inappropriate explanation of the meaning of congruence. •Unable to state minimal conditions for triangle congruence. •Makes several errors in dividing grid into congruent halves.	•Appropriate explanation of the meaning of congruence. •Understands that 3 measurements are needed for triangle congruence. •Shows at least 3 ways to divide the grid into congruent halves.	•Appropriate explanation of the meaning of congruence. •Able to give at least one set of minimal conditions for triangle congruence. •Shows at least 3 ways to divide the grid into congruent halves.	In addition to Level 3, •Provides more than one set of minimal conditions for triangle congruence. •Recognizes that not all faces of Cinesphere are congruent and provides an explanation. •Shows at least 5 ways to divide the grid into congruent halves.

Achievement Levels Defined by the Ministry of Education and Training

Level 1 Identifies achievement that falls much below the provincial standard.

Level 2 Identifies achievement that approaches the standard.

Level 3 The "provincial standard," identifies a high level of achievement of provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.

Level 4 Identifies achievement that surpasses the standard.

WHAT YOU MIGHT SEE

Problem Solving: Estimation By Organized Counting

Level 1

d) 10 000, it's a very large structure
but it can't be too big
30 000, e) 25 000

This student appears to have made a holistic estimate. There is no evidence that the student used the photograph as a basis for constructing some organized method of counting and estimating. The estimate of 10 000 is well beyond the limits of any reasonable estimate (800-2400).

Level 2

d) 38 vertical sectors and 14 horizontal parallel spaces.
also in one square unit there is one triangle and two halves of one (2 full triangles)

So I estimate 2
×38
76
14
304
76
about 1064 triangles

1064
× 2
2328 + 1 = 2328 struts

The student counted 38 slices of longitude which he calls "vertical sectors" and 14 slices of latitude, which he calls "horizontal parallel spaces," to obtain an estimate of the number of parallelograms formed by the struts. Then he multiplied by 2 to obtain the reasonable estimate of 1064 for the number of triangular faces.

To estimate the number of struts, he doubled the number of faces. This shows that he recognized that merely multiplying by 3 would involve double counting, but he did not discover that the number of struts is 3/2 times the number of faces. There was no estimate given for the number of vertices.

Level 3

d. I estimate there are 1600 faces on the Cinesphere.
First I estimated the number of faces around its widest point. That estimate was 40. 40x40=1600.

I estimate there are 2400 struts on Cinesphere.
e. I estimate there are 4800 vertices on Cinesphere.

The reasonable estimate of 1600 was obtained by counting number of triangles visible around the equator as 40 (actual number 45), estimating the number of rows as 40 (actual number 32), and calculating the product of 40 × 40. The student then estimated the number of struts by multiplying the estimated number of faces by 3/2 (the correct factor). She then tripled the number of faces to estimate the number of vertices (instead of dividing by 2), apparently not realizing that each vertex is shared by 6 faces.

Level 4

d) 1332 faces of triangles I got 18 when I counted the triangles that were in a row close to the "equator" of the sphere. I estimated that each row one above or below the "equator" would have 1 less triangle (on the visible side of the Cinesphere). Therefore the visible hemisphere has 666 faces, plus the opposite side = 666×2=1332 ~~so~~ 1998 struts,
e) 999 vertices.

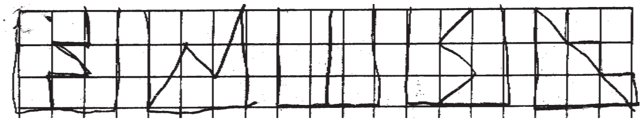
This response is a model of the kind of thinking that we need to encourage in our students. This student counted 18 upright triangles (and therefore 18 inverted triangles) for a total of 36 triangular faces around the equator of Cinesphere. He assumed that there was one fewer triangle in each successive row. Since he assumed the top would have 1 triangular face, the total number of triangular faces was estimated to be 1 + 2 + 3 + … + 36 or 36 × 37 ÷ 2 which is 666. He then doubled this to account for the hemisphere that was not visible and obtained the very good estimate of 1332. This student went on to use the correct factors, 3/2 and 1/2 to estimate the number of edges and vertices from the number of faces!

WHAT YOU MIGHT SEE

Concepts: Understanding the Concept of Congruence of 2-D Shapes

Level 2

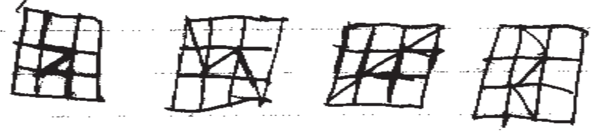
①a) it means there the exact same size and shape (position dosen'T matter).
b) put it on top of the other one and rotate it intil it gets to the right position for 360° and if it mathes exactly its congruent
c) 1 because you eather need 1 mesurment and two Degrees or 2 mesurments an 1 Degre
d) 5 see grid paper
②b) yes because there the same on each side



This response includes an appropriate definition of congruence. It also includes four distinct and correct divisions of the grid into congruent halves. The student seems to understand that three measurements are needed to establish congruence, but confuses degrees and angles.

Level 3

1. a) It means that they are the same in shape and size.
b) place them one on top of the other to see if they are the same
c) 1 edge and two angles.
d) 4 different ways.

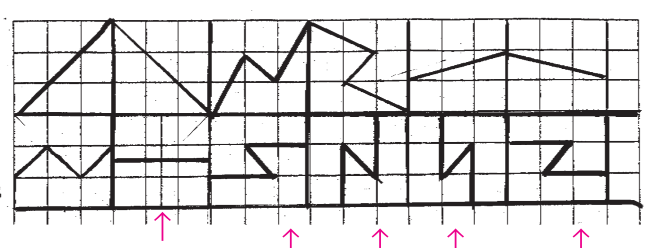


This student demonstrates an understanding of the meaning of congruence and correctly recognizes that the triangles on the surface of Cinesphere are not all congruent. He also offers "one edge and two angles" as a set of minimal conditions for congruence. This is essentially correct as long as the two equal sides are corresponding sides. Finally the student displays four distinct and correct partitions of the grid into congruent halves.

2. a) triangle, rhumbus, hexagon, trapazoid, pentagon.
b) No, I think that they get smaller towards the top in order to close off the sphere more effectivly.

Level 4

1a. 'Congruent' means that two polygons are exactly the same size and shape.
b. You can test whether two polygons are congruent by placing one on top of the other.
c. Three: lengths of all three sides or lengths of two sides and an angle or length of one side and two angles
2a. triangles, rhombus, parallelograms, hexagons, quadrilaterals
b. No, because if they were the would form a 2-D tessellation instead of a 3-D structure



This one is not valid because its vertices are not grid points.

These are transformed images of each other

The response to 1 c) includes three distinct sets of minimal conditions for triangle congruence. The student has also displayed five distinct ways of dividing the 3 × 3 grid into congruent halves. (The top row displays six partitions of the grid, but only three of them are distinct and the other three are transformed, i.e. rotated or reflected, images.) The student also realizes that the triangular faces on the surface of Cinesphere are not all congruent and attempts an explanation. Although the explanation is not clear, it suggests that the student understands that a 2-D tessellation of congruent triangles on a flat surface does not map into a tessellation of congruent triangles on the surface of a sphere.

Activity 2 – Teacher Edition

Structures from the Earliest Civilizations

Expectations Addressed

- G 7-1 recognize the front, side, and back views of three-dimensional figures.
- G 7-2 sketch front, top, and side views of three-dimensional figures with or without the use of a computer application.
- G 7-3 sketch three-dimensional objects from models and drawings.
- G 7-4 build three-dimensional figures and objects from nets.
- G 7-9 recognize the image of a two-dimensional shape under a translation, a reflection, and a rotation.

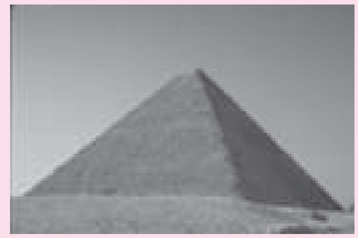
Context

As noted on page 25, Exercises ❸ and ❹ in Activity 1 are quite challenging for Grade 7 students because of their limited experience with visualization. To develop their ability to visualize three-dimensional figures, it is important to provide students with numerous opportunities to build models of such figures and manipulate the models to view them from a variety of perspectives.

Exercises ❶ and ❷ of this activity build on the students' visualization experiences in Activity 1 by having them build models of pyramids and prisms and sketch them from various perspectives. In Exercise ❸, students are given shadows and asked to identify the polyhedra that created them. Exercise ❹ is a challenge to the student's ability to visualize how forms change under rotations in space. Some students will find ingenious ways to determine which letters on the cube are on opposite faces. Some will even sketch letters in the correct orientation. Exercise ❻ uses a net (see template p. 15) for the lettered cube. By building the lettered cube from this net, students can verify their answers to Exercises ❹ and ❺, and experience how the orientations of the letters change as the cube is rotated in three dimensions. Finally, Exercise ❼ has students visualize various prisms and pyramids from diagrams. By enumerating their edges, faces, and vertices, they discover Euler's Theorem (see p. 14) for these figures.

Activity 2 – Student Page

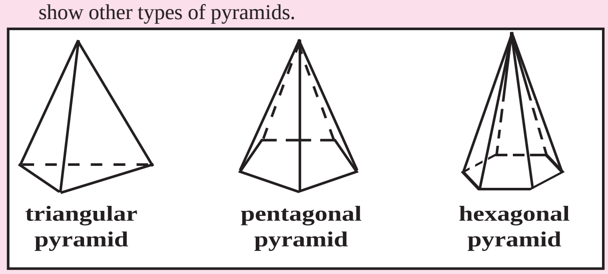
Structures from the Earliest Civilizations



The great pyramids of ancient Egypt are evidence that humans have explored the properties of shapes for thousands of years. Knowing these properties helped them build structures for living and for dying. The Pharoahs constructed these pyramids as tombs where they planned to spend their afterlife. The construction of such collossal structures required careful planning, measurement, and calculation.

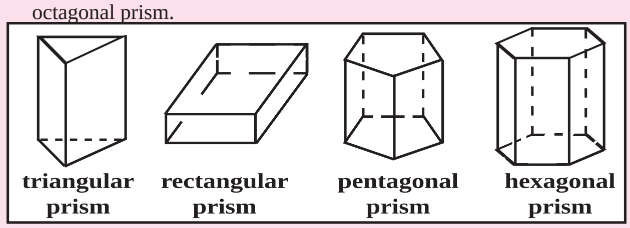
When it was completed around 2580 B. C., the Great Pyramid of Cheops (Khufu) was the tallest structure in the world. It was 146 m tall and had a square base measuring 230 m per side. (That's the area of about 8 football fields.) The huge blocks from which the pyramids were constructed were so well fitted that a piece of tissue paper can not be squeezed between them!

The Pyramid of Cheops is a square-based pyramid. The diagrams show other types of pyramids.



Study the diagrams above. Then describe how all pyramids are alike. Sketch an octagonal pyramid.

The diagrams below show different kinds of *prisms*. Study the diagrams and then describe how all prisms are alike. Sketch an octagonal prism.



Pyramids and prisms are examples of *polyhedra*. A polyhedron is any solid that has only flat faces. Hence, cylinders, cones and spheres are not polyhedra. Can you sketch a polyhedron that is not a prism or pyramid?

Activity 2 – Teacher Edition

The Lesson Launch

5 minutes

At the conclusion of Activity 1, two important questions were posed:

- How can we check whether our visualization of the truncated cube and cuboctahedron yielded a correct count of the edges, faces, and vertices?
- Is the number of edges and/or the number of vertices related to the number of faces of a three-dimensional figure?

To launch this lesson, ask a member of each pair (from Activity 1) to record on the blackboard or overhead projector the number of faces, vertices, and edges obtained for the truncated cube in Exercise ❸ of Activity 1. Have each group display its truncated cube made from the template on page 19. Explain that a model is an excellent way to check your answers to questions involving three-dimensional figures. Place a truncated cube on the overhead projector to show the silhouette from various perspectives. (Keep the projector close to the screen to avoid distortions from keystoning.) Explain how removing one vertex of the cube and replacing it with an equilateral triangle increases the number of faces by one, vertices by two, and edges by three.

Invite any student who constructed a cuboctahedron from the template to display it. Repeat for the cuboctahedron the sequence of questions and demonstrations described above. (See the template for the cuboctahedron p. 20.) Ensure that students know how to enumerate the faces, edges, and vertices of the cuboctahedron and pass around the model so they can verify their numbers.

Initiating Activity

10 minutes

Divide students into groups of three. Distribute page 34 to each student. Provide each group with 23 marshmallows and 37 toothpicks and/or straws and pipe-cleaners. Assign Exercises ❶ to ❸ to each group. When the groups are finished, have them compare their sketches and models.

Individual Activity

25 minutes

Assign Exercises ❹ and ❺ to each student. Ensure that students know what is meant by the word "perspectives" in Exercise ❹. When the students have completed these exercises, distribute to each a pair of scissors, a copy of the template on page 15, and some tape. Have them construct a model of the lettered cube to check their answers.

Activity 2 – Student Page

Structures from the Earliest Civilizations

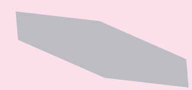
❶ Using marshmallows and toothpicks, or straws and pipe cleaners, construct models of these polyhedra.
a) triangular pyramid b) octagonal pyramid
c) pentagonal prism

❷ Suppose a pentagonal pyramid is standing on its pentagonal face.
a) Sketch how it would look from above and from the side.
b) Sketch three different kinds of shadow it could cast.

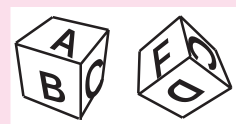
❸ Name as many polyhedra as you can that could cast each of these shadows.

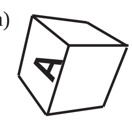
a) 

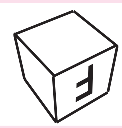
b) 

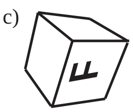
c) 

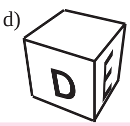
❹ The six faces of a cube are lettered from A through F. The cube is shown here from two different perspectives. Draw the missing letters the way they would appear on the cube when seen from these perspectives.



a) 

b) 

c) 

d) 

❺ Faces of a cube that share a common edge are called *adjacent*. Faces that do not share a common edge are called *opposite*. For the cube in Exercise ❹, write the letter that is on the face opposite the letter i) A ii) B iii) C. Explain how you determined what letters were opposite.

❻ Get the net for the lettered cube from your teacher. Cut out the net and fold along the dotted lines to form a cube. Tape the tabs. Then use the cube to check your answers in Exercise ❹ and ❺.

❼ Use the diagrams on the other sheet to help you count the number of faces, edges, and vertices on each of these polyhedra.
a) Complete this table.
b) Describe the relationship you find relating F, V, and E. This pattern is called *Euler's Theorem*.
c) Can you find any other relationships?

Polyhedron	Number of Faces F	Number of Vertices V	Number of Edges E
triangular pyramid			
pentagonal pyramid			
hexagonal pyramid			
triangular prism			
rectangular prism			
pentagonal prism			
hexagonal prism			

Closure

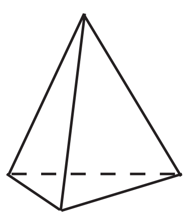
Ask students to use their models to enumerate the faces, vertices, and edges of the triangular pyramid and the pentagonal prism. Have them enter these numbers in the table on page 35. For homework, assign the task of completing the table and conjecturing a relationship among the variables F, V, and E. If your students have already studied some algebra, see the extension on page 37.

STRUCTURES FROM THE EARLIEST CIVILIZATIONS

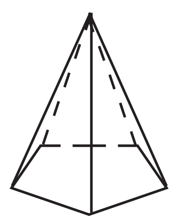
The great pyramids of ancient Egypt are evidence that humans have explored the properties of shapes for thousands of years. Knowing these properties helped them build structures for living and for dying. The Pharoahs constructed these pyramids as tombs where they planned to spend their afterlife. The construction of such collossal structures required careful planning, measurement, and calculation.

When it was completed around 2580 B. C., the Great Pyramid of Cheops (Khufu) was the tallest structure in the world. It was 146 m tall and had a square base measuring 230 m per side. (That's the area of about 8 football fields.) The huge blocks from which the pyramids were constructed were so well fitted that a piece of tissue paper cannot be squeezed between them!

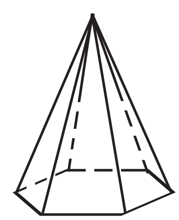
The Pyramid of Cheops is a square-based pyramid. The diagrams show other types of pyramids.



triangular pyramid



pentagonal pyramid

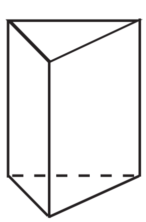


hexagonal pyramid

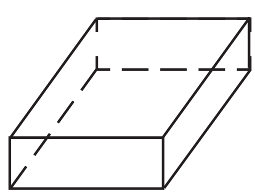
Study the diagrams above. Then describe how all pyramids are alike. Sketch an octagonal pyramid.

Note:
The dotted lines are edges that would not be visible

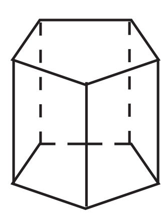
The diagrams below show different kinds of *prisms*. Study the diagrams and then describe how all prisms are alike. Sketch an octagonal prism.



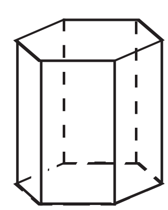
triangular prism



rectangular prism



pentagonal prism



hexagonal prism

Pyramids and prisms are examples of *polyhedra*. A polyhedron is any solid that has only flat faces. Hence, cylinders, cones and spheres are not polyhedra. Can you sketch a polyhedron that is not a prism or pyramid?

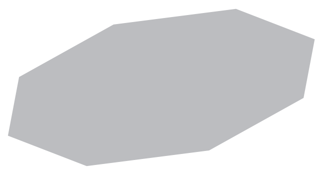
STRUCTURES FROM THE EARLIEST CIVILIZATIONS

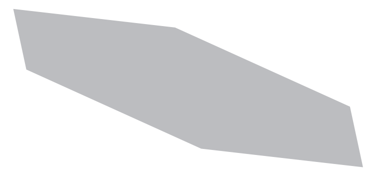
❶ Using marshmallows and toothpicks, or straws and pipe cleaners, construct models of these polyhedra.
a) triangular pyramid b) octagonal pyramid c) pentagonal prism

❷ Suppose a pentagonal pyramid is standing on its pentagonal face.
a) Sketch how it would look from above and from the side.
b) Sketch three different kinds of shadow it could cast.

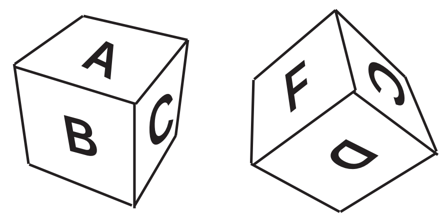
❸ Name as many polyhedra as you can that could cast each of these shadows.

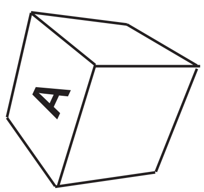
a)

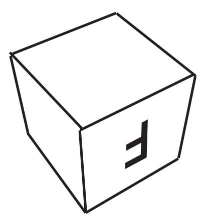

b)


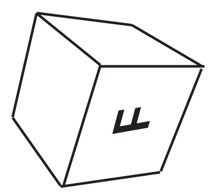
c)


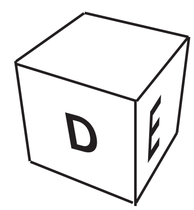
❹ The six faces of a cube are lettered from A through F. The cube is shown here from two different perspectives. Draw the missing letters the way they would appear on the cube when seen from these perspectives.



a)


b)


c)


d)


❺ Faces of a cube that share a common edge are called *adjacent*. Faces that do not share a common edge are called *opposite*. For the cube in Exercise ❹ , write the letter that is on the face opposite the letter i) A ii) B iii) C. Explain how you determined what letters were opposite.

❻ Get the net for the lettered cube from your teacher. Cut out the net and fold along the dotted lines to form a cube. Tape the tabs. Then use the cube to check your answers in Exercise ❹ and ❺.

❼ Use the diagrams on the other sheet to help you count the number of faces, edges, and vertices on each of these polyhedra.
a) Complete this table.
b) Describe the relationship you find relating F, V, and E. This pattern is called *Euler's Theorem*.
c) Can you find any other relationships?

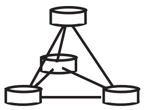
Polyhedron	Number of Faces F	Number of Vertices V	Number of Edges E
triangular pyramid			
pentagonal pyramid			
hexagonal pyramid			
triangular prism			
rectangular prism			
pentagonal prism			
hexagonal prism			

Grade 7

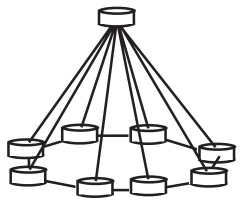
Answer Key for Activity 2

❶ The models should look something like these.

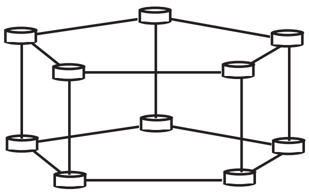
a) triangular pyramid



b) octagonal pyramid

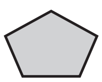


c) pentagonal prism

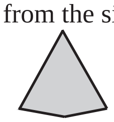


❷ a) Two perspectives on a pentagonal pyramid.

from above



from the side



b) Answers will vary. In addition to the shadows in ❷a), we have shadows such as these.

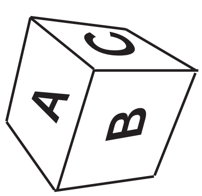
❸ The most obvious answers we anticipate from students are:

a) pentagonal prism and pentagonal pyramid.
b) octagonal prism and octagonal pyramid.
c) hexagonal prism and hexagonal pyramid.

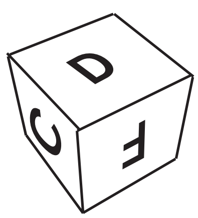
Some students may discover less obvious answers, e.g., the shadow in ❸ c) could also be produced by a cube.

❹ The cube is shown below from each perspective.

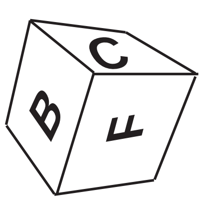
a)



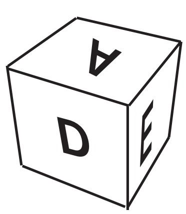
b)



c)



d)



❹ (cont'd) Many students will have difficulty visualizing the orientations of the letters as the cubes are rotated. To facilitate this visualization, have students imagine each letter in standard, i.e., upright position and have them sketch the letters on the adjacent faces. For example, when B is in standard position, the diagram shows that A is on the face above it with its "feet" pointing toward B, and C is in standard position on an adjacent face to the right with its open end away from B. When students have sketched the six faces with each letter in standard position, they find it easy to visualize the orientations of the letters under rotations.

❺ (i) F (ii) D (iii) E

Answers will vary. To determine what face is opposite A, a student might look at the first diagram in ❹ and observe that when C is in standard position, the face with A is directly above it and the face opposite A is directly below it. The second diagram in ❹ shows that the face below C is F, so F is opposite A.

❼ a) The completed table is shown below.

Polyhedron	Number of Faces F	Number of Vertices V	Number of Edges E
triangular pyramid	4	4	6
pentagonal pyramid	6	6	10
hexagonal pyramid	7	7	12
triangular prism	5	6	9
rectangular prism	6	8	12
pentagonal prism	7	10	15
hexagonal prism	8	12	18

b) We observe that for each three-dimensional figure in the table F + V – E = 2. This relationship is known as *Euler's Theorem*. That is, for all polyhedra, the number of faces plus vertices is two more than the number of edges.

c) Students might discover other relationships such as:

- for pyramids, F = V.
- for prisms, as the order of the polygonal base increases by one, the number of faces increases by one, the number of vertices by two and the number of edges by three.

The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1- 8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 2

	Level 1	Level 2	Level 3	Level 4
APPLICATION	•The sketches of shadows and/or the identification of polyhedra that cast given shadows have serious errors.	•At least two different and correct shadows are sketched for the pentagonal pyramid.	•At least two different and correct shadows are sketched for the pentagonal pyramid.	In addition to Level 3, •At least three distinctly different and correct shadows are sketched for the pentagonal pyramid.
Visualize and Sketch 2-D Shapes & 3-D Figures (exercise ❷ – ❺) G 7-1, G 7-2, G 7-4, G 7-9	•There are a few incorrect letters on the faces of the cubes sketched in ❹ and they are seldom drawn in the correct orientation.	•The correct letters on all faces of the cubes in ❹ are drawn and almost always in the correct orientation.	•The correct letters on all faces of the cubes in ❹ are drawn in the correct orientation. •At least one correct polyhedron is named for each shadow.	•More than one correct polyhedron is named for each shadow.

CHALLENGE

Deducing Euler's Theorem for Prisms and Pyramids

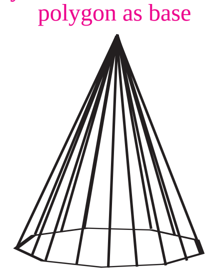
If your students have already studied some algebra, ask them to study the table they completed in Activity 2. Prompt them to generalize the patterns in the table to a pyramid with a polygonal base of *n* sides. Show a picture like the one on the right and ask students to write expressions (in terms of *n*) for the number of its faces, vertices, and edges. Ask students to check whether Euler's Theorem holds for such a pyramid. Ask them whether this means that Euler's Theorem is true for all pyramids and encourage them to justify their answers.

(Answer: F = n + 1, V = n + 1, and E = 2n. Since (n+ 1) + (n + 1) = 2n + 2, then F + V = E + 2, and Euler's Theorem is satisfied for all pyramids.)

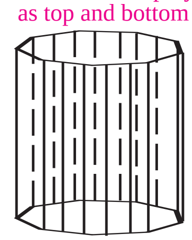
Repeat the same sequence of questions for a prism having congruent polygons of *n* sides forming its top and bottom.

(Answer: F = n + 2, V = 2n, and E = 3n. Since (n+ 2) + 2n = 3n + 2, then F + V = E + 2, and Euler's Theorem is satisfied for all prisms.)

Pyramid with an *n*-sided polygon as base



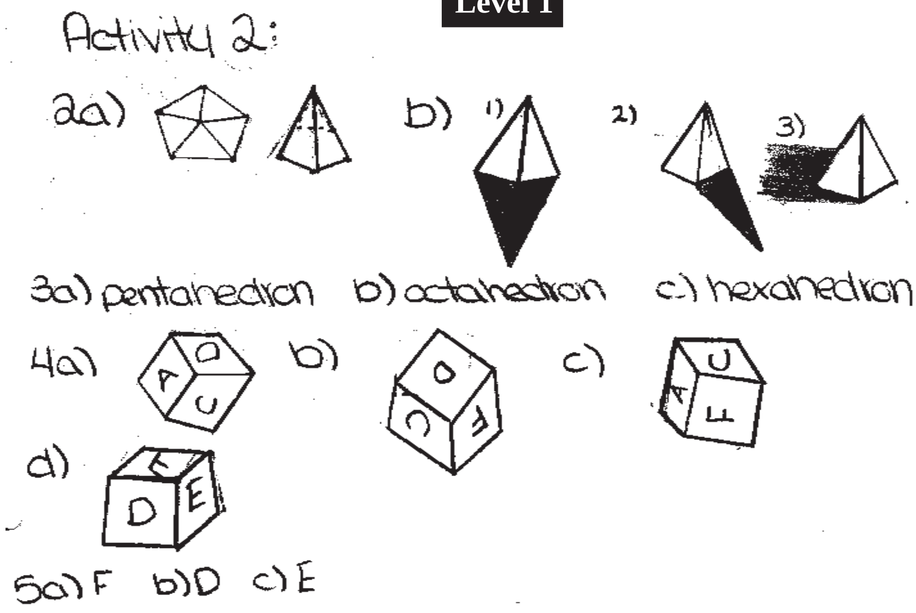
Prism with *n*-sided polygons as top and bottom



WHAT YOU MIGHT SEE

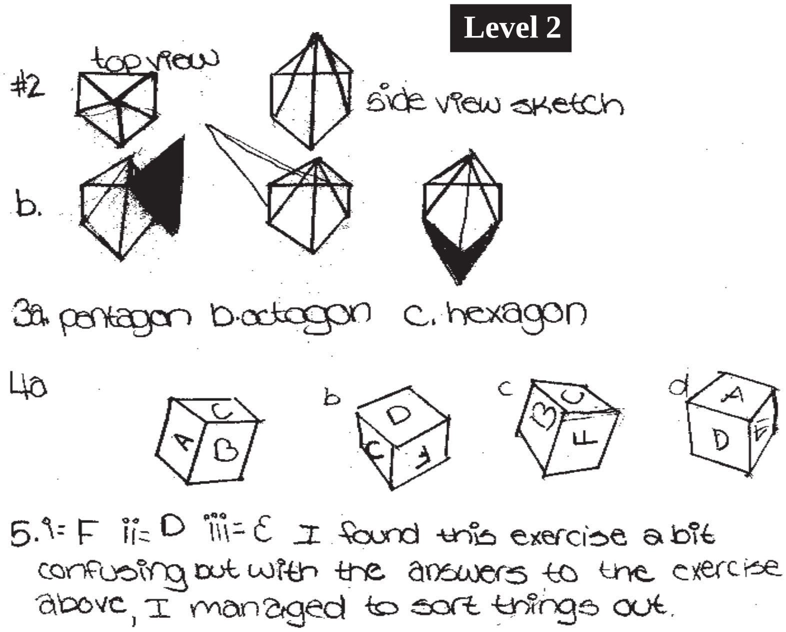
Application of Mathematical Procedures: Visualize and Sketch 2-D Shapes & 3-D Figures

Level 1



The shadows of the pentagonal pyramid are drawn correctly. The solids named in Exercise ❸ exist, but it is not clear whether the student knows what solids these describe. (If a chat with the student reveals that she can sketch or describe these solids, then it would be appropriate to revise this upward to Level 2.) The student has identified correctly the pairs of opposite sides in Exercise ❺. However, the cubes drawn in Exercise ❹ have incorrect letters in several cases and often in the wrong orientation. The student does not appear to have assembled the lettered template to check her answers.

Level 2

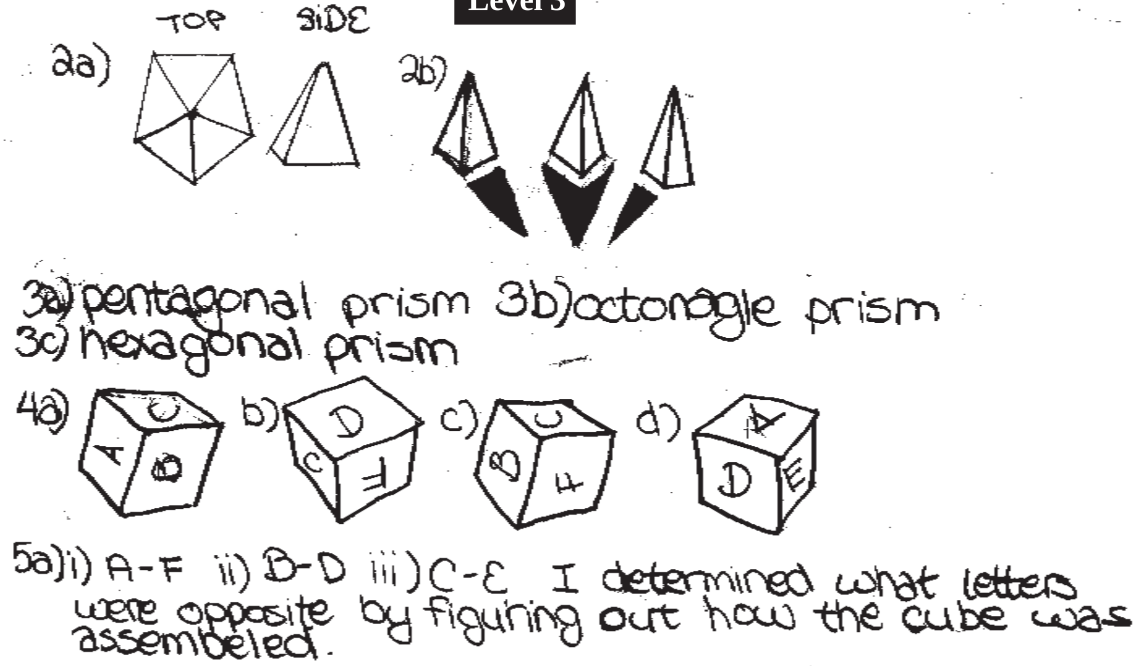


The shadows of the pentagonal pyramid are drawn correctly. The names given in Exercise ❸ are names of polygons rather than polyhedra. The cubes drawn in Exercise ❹ have correct letters in all cases and almost always in the correct orientation. The student appears to have discovered how to determine opposite sides by referring to the answers in Exercise ❹.

WHAT YOU MIGHT SEE

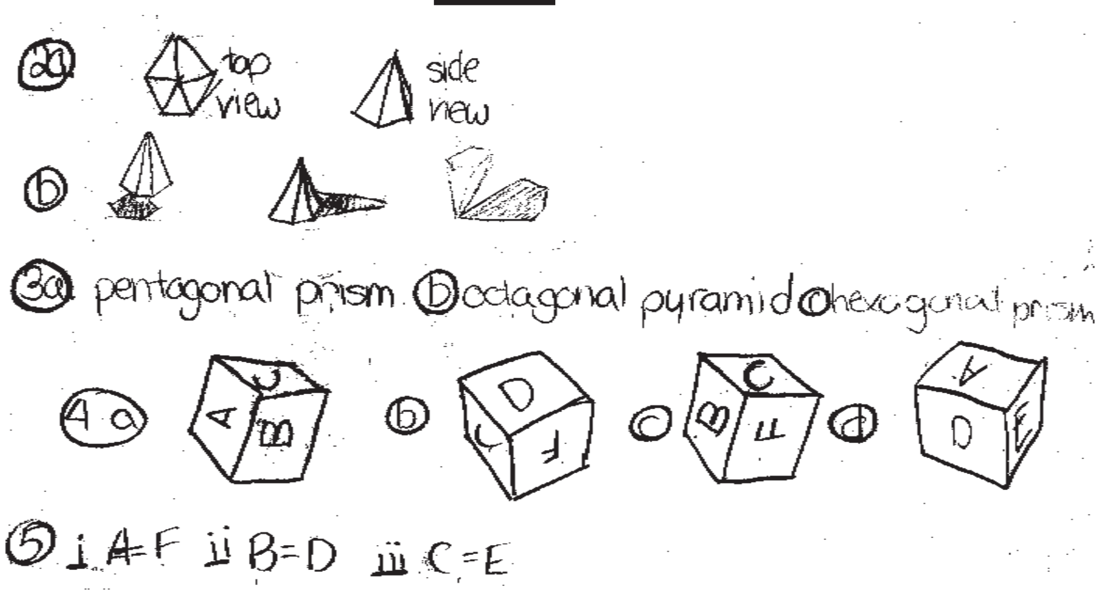
Application of Mathematical Procedures: Visualize and Sketch 2-D Shapes & 3-D Figures

Level 3



The shadows of the pentagonal pyramid are drawn correctly. Appropriate solids are named in Exercise ❸. The student has identified correctly the pairs of opposite sides in Exercise ❺. The cubes drawn in Exercise ❹ have correct letters in the correct orientation. (The C in ❹ b & ❹ c should be rotated a little.)

Level 3



The shadows of the pentagonal pyramid are drawn correctly. The student has even shown the shadow that would be cast by an inverted pentagonal pyramid. Appropriate solids are named in Exercise ❸. The student has identified correctly the pairs of opposite sides in Exercise ❺. The cubes drawn in Exercise ❹ have correct letters in the correct orientation.

Activity 3 – Teacher Edition

The Platonic Solids of Ancient Greece

Expectations Addressed

G 7-4 build three-dimensional figures and objects from nets.

G 7-6 explain why two shapes are congruent.

G 7-7 identify through investigation the conditions that make two shapes congruent.

Context

This activity aims at developing visualization of three-dimensional figures through hands-on experiences. Students are introduced to the Platonic solids and are provided with templates (see pp. 16-18) they can use to construct these solids. Using their constructed models of the tetrahedron, cube, octahedron, and dodecahedron, students enumerate the faces, vertices, and edges and record these numbers in a table. By checking the numbers in their tables, students are able to verify that Euler's Theorem is valid for the first four Platonic solids.

Using the pictorial representation of the icosahedron, students must estimate the number of faces, vertices, and edges of this fifth Platonic solid. Using their estimates, they can conjecture whether Euler's theorem is valid for the icosahedron. The process of estimating the number of faces of an icosahedron from a two-dimensional representation is reminiscent of the estimation of the faces of Cinesphere from a photo and involves the same kind of visualization skills. To verify their estimates, students are then provided with a template (see p.18) for an icosahedron. By cutting out and taping this template together, students construct a model of the icosahedron they can use to enumerate its faces, edges, and vertices. Upon recording these values in the table, they can verify that the icosahedron also satisfies Euler's Theorem. Alternatively, students may find it too difficult to count all the edges of the icosahedron and are prompted to deduce this number by the following reasoning.

It is clear from the template that the icosahedron has 20 faces. Since each face has 3 edges, we might expect a total of 3×20 or 60 edges. However, each edge borders two faces and is therefore counted twice, so we must divide by 2 to obtain the actual number of edges, i.e., 30. Similarly the number of vertices is $60 \div 5$, or 12 because each vertex is shared by 5 faces and is therefore counted five times.

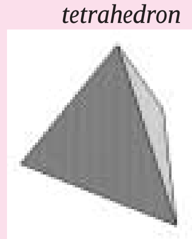
Activity 3 – Student Page

The Platonic Solids of Ancient Greece

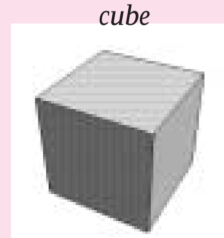
…mathematicians are really seeking to behold the things themselves, which can be seen only with the eye of the mind.
– Plato (c. 427-347 B.C.)

About 2500 years ago, the great Greek philosopher Plato became fascinated with the geometric properties of *regular polyhedra.* (A regular polyhedron has all faces congruent and each face is a polygon with sides of equal length.) The ancient Greek mathematicians had discovered that there are exactly five regular polyhedra. The beauty and symmetry of these geometric forms prompted him to associate four of these regular polyhedra with the four elements that they believed to compose all things. He associated the tetrahedron with fire, the cube with earth, the octahedron with air and the icosahedron with water. The fifth regular polyhedron, the dodecahedron, was thought to symbolize the universe. These five regular polyhedra are usually called *Platonic Solids* as a tribute to Plato.

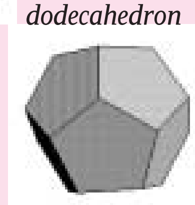
tetrahedron



cube

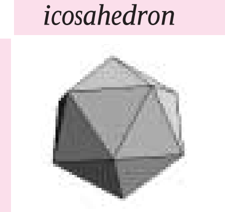


dodecahedron



octahedron

icosahedron



What is a *regular polyhedron?*

What regular polygons form the faces of each of the Platonic solids?

Activity 3 – Teacher Edition

The Lesson Launch 10 minutes

Launch this lesson with questions such as the following to revisit concepts introduced in the previous activities.

- *What is a polyhedron? Give two examples.*
- *Name two three-dimensional figures that are not polyhedra.*
- *What is a "face" of a polyhedron; a "vertex"? an "edge"?*
- *How many faces of a polyhedron share the same edge?*
- *State Euler's Theorem and explain what it means.*
- *Do the truncated cube and the cuboctahedron satisfy Euler's Theorem?*

Hold up models of a tetrahedron, cube, and octahedron from a set of geometric solids. Ask students to name each three-dimensional figure. Have students observe that all the faces of such figures are congruent. Ask students to guess how many polyhedra of different shape exist that have all sides congruent.

Initiating Activity 5 minutes

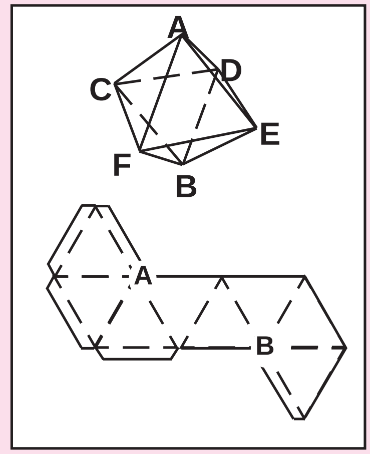
Distribute page 42 to the students. Ask them to read that page and then ask them to describe what is meant by a *Platonic solid.* Have them describe in turn the properties of each of these solids. Ask them whether there are any prisms that are Platonic solids. (Ans: only the cube). Ask whether there are any pyramids that are Platonic solids. (Ans: only the triangular pyramid) Ensure that students understand that prisms and pyramids are not generally Platonic solids because not all faces of prisms and pyramids are the same shape as the base. Ask whether students are surprised to learn that there are only five Platonic solids.

Distribute page 43 and a copy of the template on page 16 for the octahedron. Provide students with scissors and tape and have them attempt Exercise ❶ on their own.

Individual Activity 35 minutes

Check that all students were able to construct the octahedron from the template. Then provide them with the templates on pages 17 and 18 and assign them Exercises ❷ through ❺ to work on individually. Remind them that they are to cut along solid lines on the template and fold along the dotted lines. Circulate around the room helping students who have fine motor problems to assemble and tape the models.

Activity 3 – Student Page



❶ a) The octahedron shown here has its vertices labelled. Label the vertices of the net to match. Two of the vertices have been labelled for you.
b) Get the net for the octahedron from your teacher. Label its vertices as in Part a). Cut out the net and fold along the dotted lines. Tape the tabs to form an octahedron. Use the octahedron to check your answer in Part a).

❷ Get the net for the dodecahedron from your teacher. Cut out the net and fold along the dotted lines to form a dodecahedron. Tape the tabs.
a) What shape are the faces?Are they all congruent? Explain how you know.
b) Measure all the angles on any of the faces. Are all the angles congruent?

❸ The tetrahedron has four faces, all of which are congruent, equilateral triangles. Draw a net for a tetrahedron. Then cut it out and fold it to form a tetrahedron.

❹ a) Complete this table showing the number of edges, faces, and vertices of the tetrahedron, the cube, the octahedron, and the dodecahedron.

Platonic Solid	Number of Faces	Number of Vertices	Number of Edges
tetrahedron			
cube			
octahedron			
dodecahedron			
icosahedron			

b) Is Euler's Theorem true for the first four Platonic solids?
c) Using only the diagram on page 42, try to visualize an icosahedron. Estimate how many faces it has and write this number (in pencil) in your table.
d) Use your estimate in Part c) to estimate the number of edges and vertices. Explain how you obtained your estimates.

❺ a) Get the net for the icosahedron from your teacher. Use the net to determine the actual number of faces on an icosahedron. Can you determine the actual number of edges from this information by observing that every edge joins two faces and every face has three edges?
b) Do all the Platonic solids satisfy Euler's Theorem? Explain why or why not.

Closure

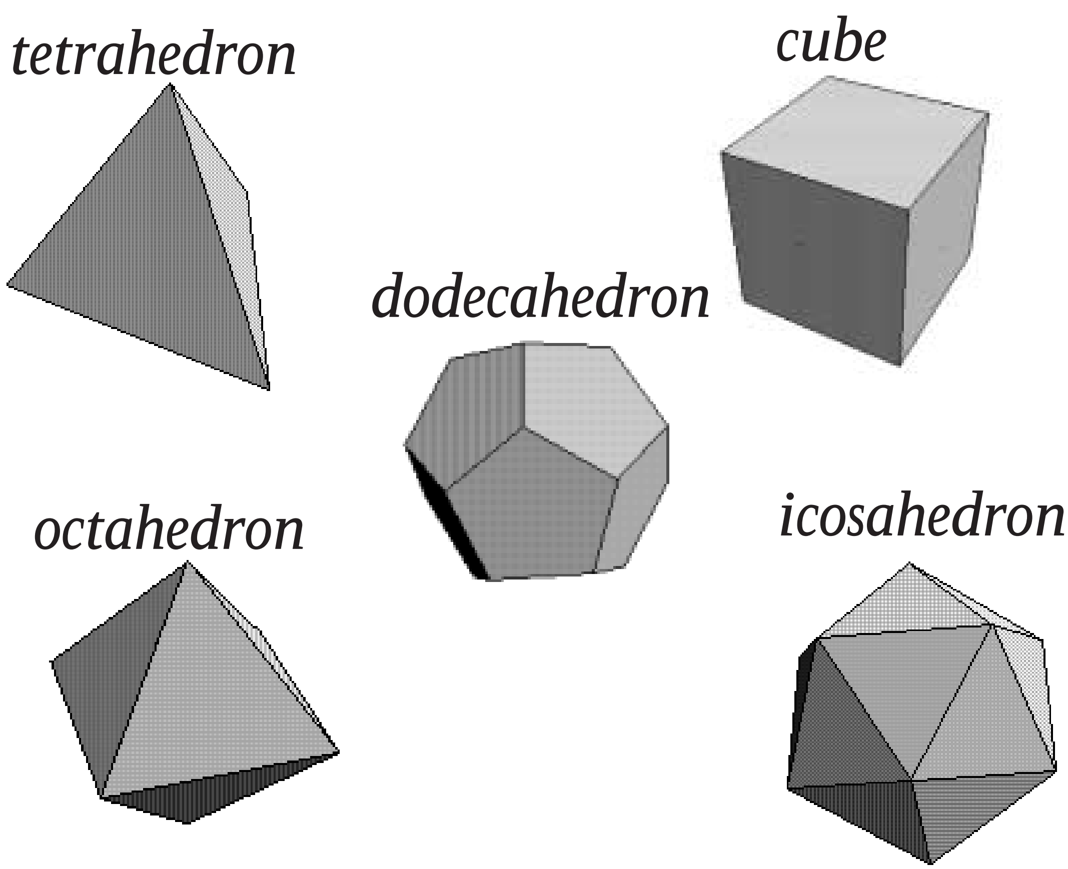
As most students complete their models, check their tables to ensure that their enumeration of faces, vertices, and edges is correct. Ask how many students had difficulty counting edges and vertices of the icosahedron. Discuss how you could use Euler's Theorem to calculate the number of edges if you knew the number of faces and vertices. Show students how they could deduce the number of vertices and edges of an icosahedron knowing only that it has 20 faces, by using the counting procedure described in the last paragraph of page 40.

Activity 3 – Student Page

The Platonic Solids of Ancient Greece

…mathematicians are really seeking to
behold the things themselves, which can be
seen only with the eye of the mind.
– Plato (c. 427-347 B.C.)

About 2500 years ago, the great Greek philosopher Plato became fascinated with the geometric properties of *regular polyhedra.* (A regular polyhedron has all faces congruent and each face is a polygon with sides of equal length.) The ancient Greek mathematicians had discovered that there are exactly five regular polyhedra. The beauty and symmetry of these geometric forms prompted him to associate four of these regular polyhedra with the four elements that they believed to compose all things. He associated the tetrahedron with fire, the cube with earth, the octahedron with air and the icosahedron with water. The fifth regular polyhedron, the dodecahedron, was thought to symbolize the universe. These five regular polyhedra are usually called *Platonic Solids* as a tribute to Plato.

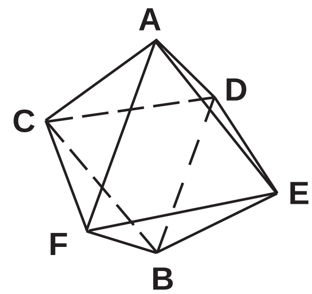


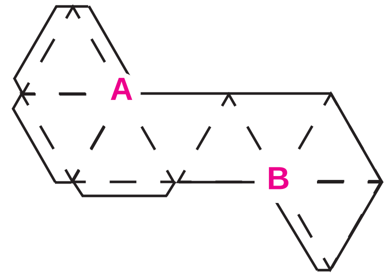
What is a *regular polyhedron?*

What regular polygons form the faces of each of the Platonic solids?

The Platonic Solids of Ancient Greece

❶ a) The octahedron shown here has its vertices labelled. Label the vertices of the net to match. Two of the vertices have been labelled for you.





b) Get the net for the octahedron from your teacher. Label its vertices as in Part a). Cut out the net and fold along the dotted lines. Tape the tabs to form an octahedron. Use the octahedron to check your answer in Part a).

❷ Get the net for the dodecahedron from your teacher. Cut out the net and fold along the dotted lines to form a dodecahedron. Tape the tabs.
a) What shape are the faces?Are they all congruent? Explain how you know.
b) Measure all the angles on any of the faces. Are all the angles congruent?

❸ The tetrahedron has four faces, all of which are congruent, equilateral triangles. Draw a net for a tetrahedron. Then cut it out and fold it to form a tetrahedron.

❹ a) Complete this table showing the number of edges, faces, and vertices of the tetrahedron, the cube, the octahedron, and the dodecahedron.

Platonic Solid	Number of Faces	Number of Vertices	Number of Edges
tetrahedron			
cube			
octahedron			
dodecahedron			
icosahedron			

b) Is Euler's Theorem true for the first four Platonic solids?

c) Using only the diagram on page 42, try to visualize an icosahedron. Estimate how many faces it has and write this number (in pencil) in your table.

d) Use your estimate in Part c) to estimate the number of edges and vertices. Explain how you obtained your estimates.

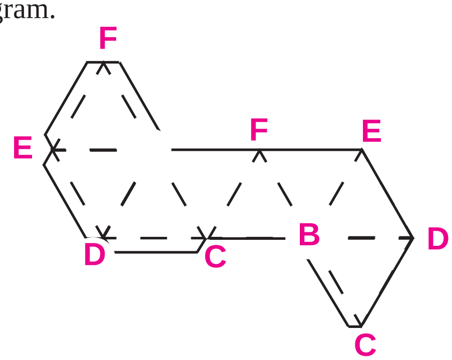
❺ a) Get the net for the icosahedron from your teacher. Use the net to determine the actual number of faces on an icosahedron. Can you determine the actual number of edges from this information by observing that every edge joins two faces and every face has three edges?

b) Do all the Platonic solids satisfy Euler's Theorem? Explain why or why not.

Grade 7

Answer Key for Activity 3

❶ If the vertices of the net are labelled as shown below, then the net, when cut out and folded, will form an octahedron with the vertices as shown in the diagram.

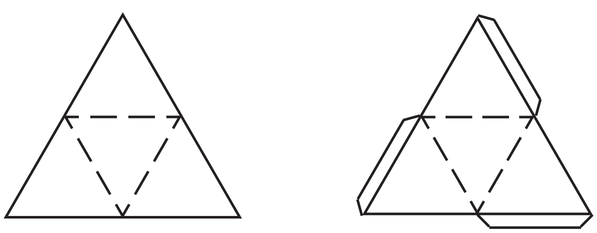


❷ a) The faces are congruent regular pentagons. The students can verify congruence by superposition, i.e., folding the template to make pairs of faces coincide.

b) All the angles have a measure of 108˚ and are therefore congruent.

❸ The diagrams show one (of several possible) tetrahedron nets with and without tabs for taping the sides.

without tabs　　　　　with tabs



Students will require some time to discover how to create a net for a tetrahedron. It helps to have them look at a model of a tetrahedron that they can use to visualize how the faces fit together.

❹ a) The completed table is shown for all 5 Platonic solids.

Platonic Solid	Number of Faces	Number of Vertices	Number of Edges
tetrahedron	4	4	6
cube	6	8	12
octahedron	8	6	12
dodecahedron	12	20	30
icosahedron	20	12	30

❹ b) Since F + V – E = 2 for the first four Platonic solids in the table in ❹ a, then Euler's theorem is true for them.

c) Students will estimate the number of faces on the icosahedron as they estimated the faces on Cinesphere, i.e., by counting the visible faces and approximating the number of hidden faces. They should be able to count at least 8 faces and obtain an estimate of between 16 and 24 faces for the icosahedron. The actual number of faces is 20.

d) Most students will approximate the number of edges and vertices by counting those which are visible and estimating the number that are hidden. A few students might try a more organized way of counting as described in the answer to ❺ a.

❺ a) The actual number of faces is 20. Each triangular face has 3 edges, so we might expect that the total number of edges is 3 times the number of faces. However, since each edge bounds 2 faces, we would be counting each edge twice. Therefore, we must divide by two. That is, the number of edges is 3/2 times the number of faces.

By similar reasoning, each face has three vertices so we might expect the number of vertices to be 3 times the number of faces. However, each vertex is shared by five faces, so we must divide by 5 to avoid counting the same vertex 5 times. That is, the number of vertices is 3/5 times the number of faces. Students will need some help with this reasoning, but their estimates should reflect some organized counting such as this.

The actual number of faces of an icosahedron is 20 so the actual number of edges is $3/2 \times 20$, or 30. The actual number of vertices is $3/5 \times 20$, or 12.

b) Students can easily verify that F = 20, V = 12 and E = 30 satisfy F + V– E = 2, and so all the Platonic solids satisfy Euler's Theorem.

The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 3

	Level 1	Level 2	Level 3	Level 4
PROBLEM SOLVING				
Using Pattern Recognition to Discover Euler's Theorem and Apply It to the Solution of Problems (exercise ❹ & ❺) G 7-4, N 7-16	•The table in Exercise ❹ a has several errors. •There is no evidence that the student understands Euler's Theorem and is able to apply it correctly.	•The table in Exercise ❹ a has minimal or no errors. •There is no evidence that the student understands Euler's Theorem and is able to apply it correctly.	•The table in Exercise ❹ a has minimal or no errors. •There is evidence that the student understands Euler's Theorem and is able to apply it correctly.	In addition to Level 3, •There is evidence of an appropriate strategy for deducing the number of edges and/or vertices of a polyhedron from its number of faces.

Coming Attraction

Build A Geodesic Clubhouse for Your Classroom

In preparation for Activity 4 of this unit, prospective teachers in course MAT 329 at the University of Toronto constructed this geodesic dome. They followed the directions on page 52 of this module and assembled the dome in 24 minutes.

In the next Activity, you will discover how a class of elementary students at Jean Vanier School in London, Ontario, using the same plans, built their own geodesic clubhouse. Try it with your students. Everyone has fun – the young and the older!

WHAT YOU MIGHT SEE

Problem Solving: Using Pattern Recognition to Discover Euler's Theorem

Level 1

4a

Platonic Solid	# of faces	# of vertecies	# of edges
Tetra-hedron	3	4	6
Cube	4	8	~~8~~ 12
Octahedron	8	16	24
Dodecahedron	12	16	50
Icosahedron	20	20	50

4 b ?

4 c. 20

4 di. 50 edges (I doubled the number of faces and then added 10)

4 dii. 50 verticies (I did the same for 4di)

This table has several errors, including at least one in each row. There is no evidence of any understanding of Euler's Theorem. In fact, the question mark in the answer to ❹ b suggests the student has not heard of this result. (This student did not complete the latter part of Activity 2.)

Level 2

4. A)

Plantonic Solid	Number of faces	Number of Vertices	Number of Edges
tetrahedron	4	4	6
cube	6	8	12
octahedron	8	6	12
dodecahedron	12	20	30
D) C) icosahedron	20	12	30

B) ~~The~~ it isn't. C) 20 D) I estimated 20 because looking at the drawing you see half, and there are 10 on the side. For the vertices ~~and~~ the edges I did the same.

5. A) I was corret. Yes, you can. B) I got 40 but ~~thats~~ not correct.

This table has no errors. In ❹ d the student indicates that Euler's Theorem is not true for the first four Platonic solids, in spite of the fact that the table entries would suggest the opposite. This indicates that the student is either unaware of the Euler relationship or unable to apply it.

WHAT YOU MIGHT SEE

Problem Solving: Using Pattern Recognition to Discover Euler's Theorem

Level 3

4a)

Platonic Solid	Number of Faces	Number of Vertices	Number of Edges
tetrahedron	4	4	6
cube	6	8	12
octahedron	8	6	12
dodecahedron	12	20	30
icosahedron	20	12	30

4b) Yes, Euler's Theorem worked for the first four Platonic Solids.

4d) I estimated the numbers by imagining what it would like if it were 3-d and then I guessed.

5a)

(F+V-E=2)

5b) 20=F, 30=E 30-20=10+2=12 vertices

This table has no errors. In the answer to ❺ b, the student writes the equation F + V – E = 2 and then writes 20 = F, 30 = E, so E – F = 30 – 20, and she adds 2 to this to obtain V = 12. This indicates that she understands Euler's Theorem and that she can apply it successfully.

Level 4

#4

PLATONIC SOLID	NUMB. OF FACES	NUMB. OF VERTICIES	NUMB. OF EDGES
TETRAHEDRON	4	4	6
CUBE	6	8	12
OCTAHEDRON	8	6	12
DODECAHEDRON	12	20	30
ISOSAHEDRON.	~~16~~ 20	~~8~~ 12	~~24~~ 30

B) Yes, it still holds.

C) 16.

D) V=8 E=24. I noticed that the tetrahedron and octahedron, which have triangular faces, seem to follow the rule: (NUMBER OF FACES x NUMBER OF SIDES OF A FACE ÷ 2 = Number of edges) since the isosahedron has triangular faces I applied the rule. After I had the number of edges I applied Euler's theorem and find the number of verticies.

5 A 20 FACES. 20 x 3 ÷ 2 = 30 EDGES. B. 12 VERTICES.

This table has no errors. In the answer to ❹ c, the student estimates the number of faces of the icosahedron to be 16 and from this makes the correct inference that if this were correct, then the number of vertices and edges would be respectively 8 and 24. Then in his answer to ❺ a, the student discovered from the net that the actual number of faces was not 16 but 20 and went back to calculate the correct number of edges from the number of faces (20 × 3 ÷ 2). The student then substituted these values for F and E into the equation F + V = E + 2, to calculate V correctly. This student work exemplifies the kind of problem solving technique that we hope to develop in all students.

Activity 4 – Teacher Edition

Building Your Own Geodesic Club House

Expectations Addressed

G 7-4 build three-dimensional figures and objects from nets.

G 7-5 identify two-dimensional shapes that meet certain criteria (e.g., an isosceles triangle with a 40° angle).

G 7-6 explain why two shapes are congruent.

G 7-7 identify through investigation the conditions that make two shapes congruent.

Activity 4 – Student Page

Building Your Own Geodesic Club House

©YES Mag: Canada's Science Magazine for Kids. Reprinted with permission.

This letter appeared in 1998 on the Internet at the address:

http://www.yesmag.bc.ca/projects/geodesic.html

It is a webpage for YES Mag, an educational publication based in Vancouver. It appeared as one of the letters under the heading, "Here are the stories we have received so far."

Dear YES Mag,
During National Science and Technology Week '96, my class of 30 students partook in a fun and challenging task. Using the idea from YES magazine, we were doing a Design/Tech unit. I thought that my group of kids would have a blast. They sure did. I separated the class into two groups. On the first day of the task, they acted as teams of "building material factory workers" – rolling up the newspaper rods. A strong leader acted as a foreman. On the second day, the whole two groups of students proceeded to act as construction crews, also with a foreman. Two complete geodesic domes were constructed in the school's library. The whole process was photographed and videotaped.
What a swell project idea. Thanks!!!

Sincerely,
Mr. Andre Lisowyk
Jean Vanier School
London, Ontario

Context

This activity brings together many of the concepts studied in the previous activities of this unit. During a span of two math classes, students work cooperatively to build a geodesic club house. Functioning as teams with differentiated responsibilities, the students, under the leadership of an appointed foreman, wrap newspaper around broom handles or dowling to form struts. By taping the struts together in accordance with the instructions, (see p. 52) students construct the frame of a geodesic dome. The instructions for assembling each frame require that students understand the meanings of the terms "decagon," "pentagon," "5-point star," "equilateral triangle," and "isosceles triangle." In constructing each geodesic structure from struts of two different lengths, students discover that not all the triangles in the structure are congruent. This is an important realization that was broached in Activity 1 when students discussed whether the faces of Cinesphere were congruent. (The geodesic clubhouse is essentially half of an "icosahedral" geodesic dome with four triangles on each of the 20 faces.)

When students have completed each structure, they crowd into its inner space and celebrate their achievement the way Mr. Lisowyk's students did in the photo above. Then they test whether Euler's Theorem applies to their structure. When students attempt to enumerate the faces, vertices, and edges of the Platonic solids as in Activity 3, they find particular difficulty with the dodecahedron and the icosahedron. However, their geodesic clubhouse is known to be composed of 35 + 30 or 65 struts (edges) and the faces and vertices are easily counted. Students can verify that Euler's Theorem applies to their clubhouse as long as they understand that the base must be regarded as a face.

Finally, in Exercise ❸ of Activity 4, students discover that the number of faces on icosahedral geodesic domes is of the form $20n^2$ where n denotes an integer. By making a table of values of $20n^2$ for various values of n, and observing that the two clubhouses together could form a dome of 80 faces, students can conjecture that the geodesic clubhouse is an icosahedral geodesic dome of frequency $n = 2$.

Activity 4 – Teacher Edition

The Lesson Launch 10 minutes

To ignite their enthusiasm, distribute page 50 to the students. Have them read the letter from Mr. Lisowyk. Explain to the students that they will be involved in the next two classes in the project described by Mr. Lisowyk. Students will be highly motivated when they discover that they will be working with their friends to construct a geodesic clubhouse. Ask them, before they begin, to estimate how many students will fit in the geodesic clubhouse. Record some of their estimates.

Cooperative Learning Activity 50 minutes

Before beginning this activity, read the *Instructions for Building a Geodesic Club House* on page 52 and collect the necessary materials. Place the newspapers in a large stack and divide the students into two (approximately) equal groups called *crews*. Distribute the doweling or broom handles among the two crews. Appoint a supervisor for each crew. Provide each supervisor with enough copies of the *Instructions for Building a Geodesic Club House* (see template p. 52) for the crew. Assign one crew the task of making 35 struts of length 71 cm. Assign the other crew the task of making 30 struts 66 cm in length. Instruct the supervisor of the crew making the longer struts,called the *senior supervisor*, to mark all the long struts with a black marker pen to distinguish them from the shorter struts.

Since there are usually more students than broom handles or doweling, it is usually effective to have students in pairs, with one rolling the newspaper and the other taping and trimming it to the required length. Circulate among the students as they roll the struts. If the students roll the newspaper too tightly they will not be able to slide the broom handle or dowling out of the newspaper. Suggest that they roll the paper more loosely and wiggle the broom handle a little to slide it out.

By the end of the first class, students should have all the struts completed. Have them place the longer struts in one pile and the shorter struts in another. Ask each supervisor to count them to ensure that they have the correct number. Have students measure a random sample from each pile to ensure that they are the correct lengths. In preparation for the next class, ask students to study the eleven steps in the instructions so they will be ready to begin work next class.

At the beginning of the second class, review the instructions step-by-step with the students to ensure that they understand the procedures. Then delegate the task of constructing a geodesic clubhouse to the senior supervisor and the class. Circulate and provide help as needed.

Activity 4 – Student Page

❶ To construct your class geodesic club house, get these materials.

- stack of newspapers
- 15 lengths of doweling or broom handles
- black marker pen
- staples and stapler
- 10 pairs of scissors
- tape measure (marked in centimetres)

Then ask your supervisor for a copy of your instructions. Follow the instructions to create your classroom geodesic club house.

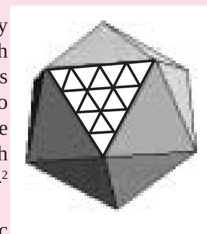
❷ When your geodesic dome is complete:

a) Count and record the number of faces, vertices and edges. Do these values of F, V, and E satisfy Euler's theorem for polyhedra? Explain why or why not.

b) Are the faces of your geodesic dome equilateral triangles? Explain how you know.

c) Are all the triangles in your geodesic dome congruent? Explain.

❸ An "icosahedral" geodesic dome is formed by building an icosahedron and then dividing each of the 20 faces of the iscosahedron into triangles as shown. Each of the 20 faces is thus divided into n^2 triangular faces, where n is some positive integer. (The diagram shows a geodesic dome with $n = 4$; i.e., each of the 20 faces is divided into 4^2 or 16 triangles.



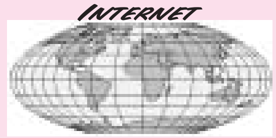
a) What are the number of faces on a geodesic dome that has each of its faces divided into n^2 faces if $n = 3$?

b) Make a table showing the possible number of faces of an icosahedral geodesic dome for various values of n.

c) Do you think your geodesic clubhouse is part of an icosahedral geodesic dome? Explain why or why not.

d) Do you think Cinesphere is an icosahedral geodesic dome? Explain why or why not.

Internet Exploration



Use the Internet address given on the other sheet to investigate the YESmag website. Can you find some other letters under the heading, "Here are the stories we have received so far"?

Report what you discover.

Here are some other websites you may wish to explore on the Internet.

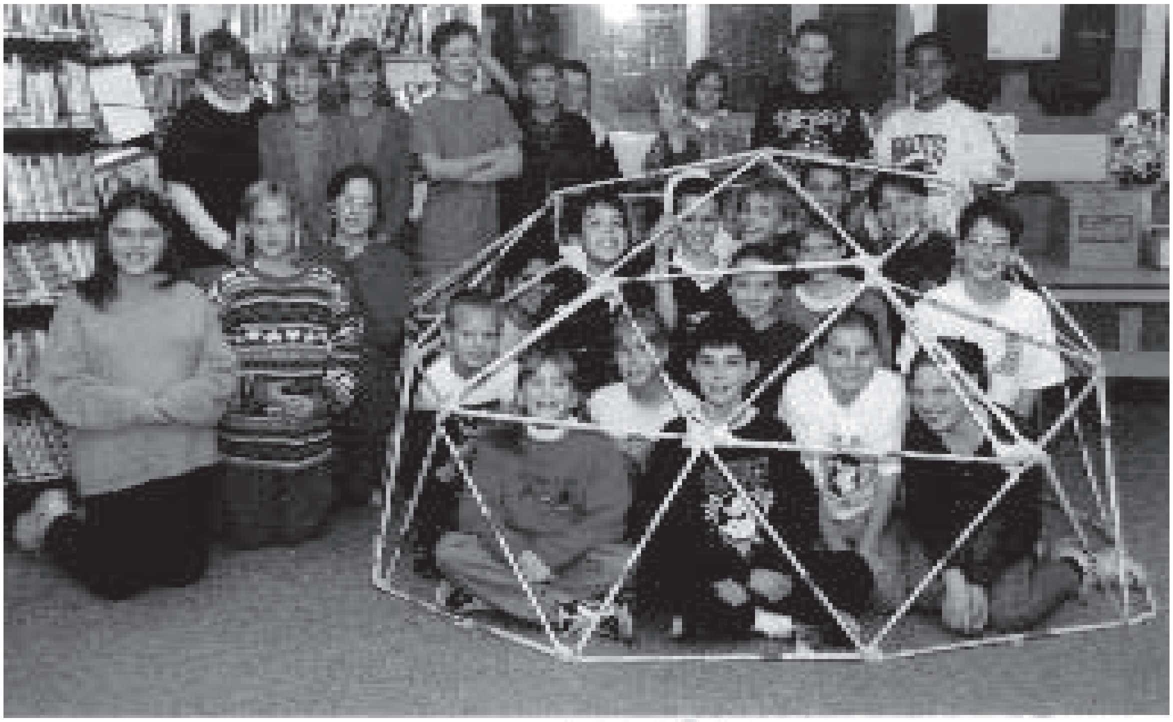
http://www.ontarioplace.com/cineshist.html

http://www.globalnow.com/nightlife/models/models.html

Closure

When the geodesic dome has been completed, invite students to crowd inside the structure to test their estimates of the number of students it could house. Distribute copies of page 51 and assign each student the task of completing Exercises ❷ and ❸. When finished, discuss these Exercises with the class.

Building Your Own Geodesic Club House



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What a swell project idea. Thanks!!!

Sincerely,
Mr. Andre Lisowyk
Jean Vanier School
London, Ontario

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a) Count and record the number of faces, vertices and edges.
Do these values of F, V, and E satisfy Euler's theorem for polyhedra? Explain why or why not.
b) Are the faces of your geodesic dome equilateral triangles? Explain how you know.
c) Are all the triangles in your geodesic dome congruent? Explain.

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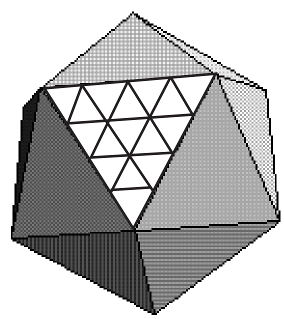
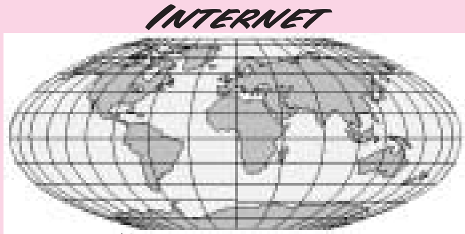


a) What are the number of faces on a geodesic dome that has each of its faces divided into n^2 faces if $n = 3$?
b) Make a table showing the possible number of faces of a geodesic dome for various values of n.
c) Do you think your geodesic clubhouse is part of an icosahedral geodesic dome? Explain why or why not.
d) Do you think Cinesphere is an icosahedral geodesic dome? Explain why or why not.

INTERNET



EXPLORATION

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Here are some other websites you may wish to explore on the Internet.

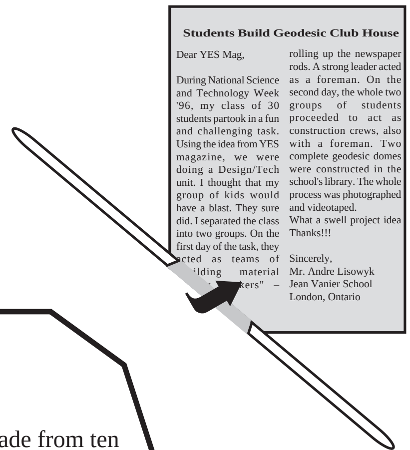
http://www.ontarioplace.com/cineshist.html
and
http://www.globalnow.com/nightlife/models/models.html

Instructions for Building a Geodesic Clubhouse

❶ Roll the broom handle inside the newspaper, from one corner to the diagonally opposite corner.

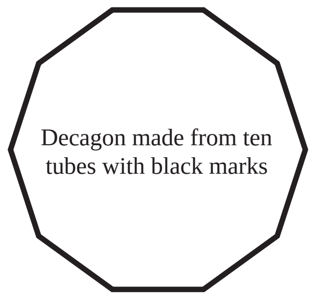
❷ Tape the newspaper so it stays rolled up. Then slide the broom handle out of the newspaper tube.

❸ Trim the ends of the newspaper tube so that it is 71 cm long. This tube is called a *strut*. Make 34 more struts this length and mark each with a black marker pen. We shall call these *black* struts.



❹ Repeat the steps above to make 30 struts each 66 cm in length. Do NOT mark these with the marker pen. We shall call these *white* struts.

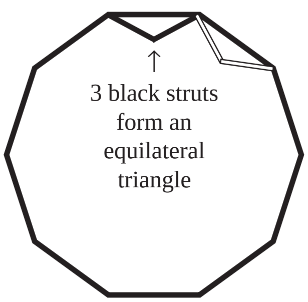
❺ Staple together 10 of the black struts to form a decagon. Adjacent tubes should overlap by 2 cm.

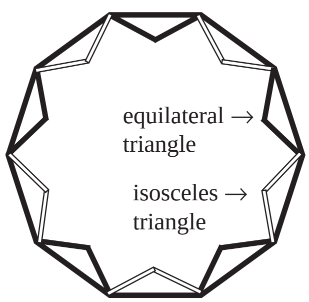


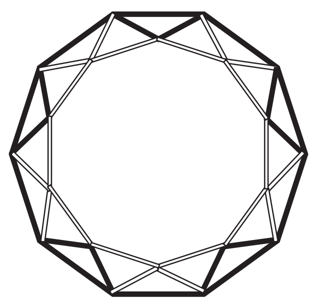
❻ Staple two black struts to one side of the decagon to form an equilateral triangle. Beside it, staple two white struts to form an isosceles triangle as shown.

❼ Repeat this pattern with alternating equilateral and isosceles triangles as shown in the diagram.

❽ Make a decagon from 10 white struts to form the inner ring and staple it to the other struts as shown.



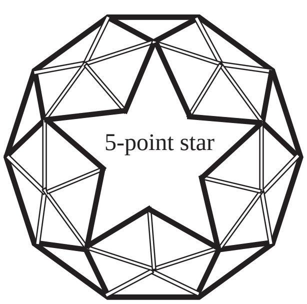


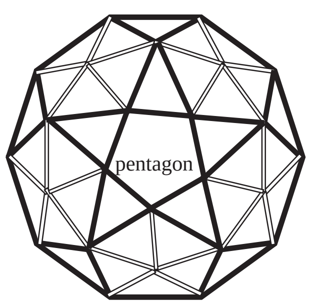


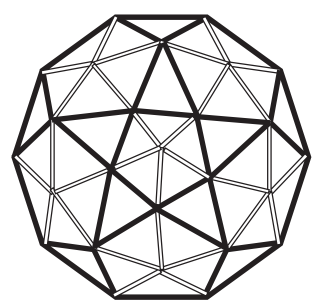
❾ Connect black struts to the vertices of the equilateral triangles to make a 5-point star. With white struts, join the other 5 vertices of the star to the vertices of the inner ring.

❿ Connect the 5 remaining black struts to the inner vertices of the 5-point star to form a pentagon at the centre as shown.

⓫ Connect the remaining 5 white struts from the centre of the pentagon to its vertices to complete your geodesic clubhouse.







Grade 7
Answer Key for Activity 4

❶ The geodesic clubhouse should look something like the picture on page 50.

❷ a) Students should know from the instructions that they used 65 tubes to form the edges. If they count the bottom of the geodesic club house as a face, they will have a total of 41 faces. They can also count to determine that there are 26 vertices. By substituting these values F = 41, V = 26, and E = 65, they can verify that F + V = E + 2 and conclude that Euler’s Theorem is satisfied. Some students may not count the bottom as a face and will assume that Euler’s Theorem fails for this structure. If this happens, remind them that Euler’s Theorem applies to closed figures.

b) The geodesic clubhouse was made from struts of two different lengths. Those triangles made from struts of the same length are equilateral, but those triangles made with two sides of different lengths are not.

c) It follows from ❷ b that since some of the triangular faces are equilateral and others are not, then not all the faces are congruent.

❸ a) If each of the 20 faces of the icosahedron is divided into n^2 faces, where $n = 3$, then there are a total of 20×9 or 180 faces.

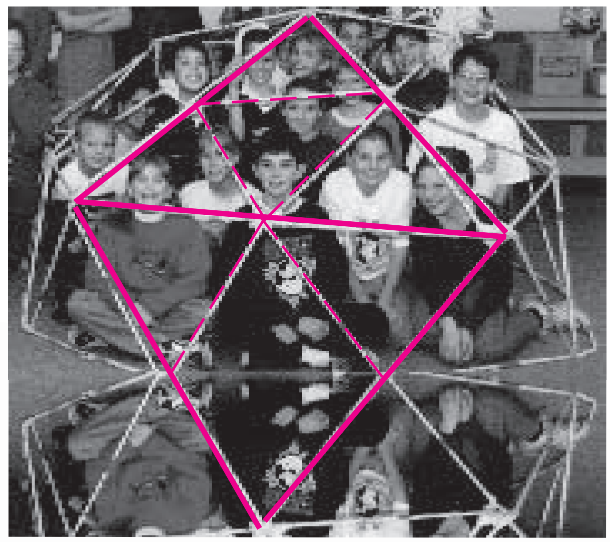
b) This table shows the possible number of faces of a geodesic dome for values of n up to $n = 10$.

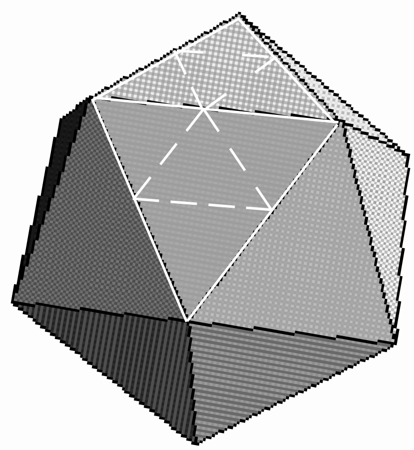
n	Number of Faces $20 \times n^2$
1	20
2	80
3	180
4	320
5	500
6	720
7	980
8	1280
9	1620
10	2000

c) If the two geodesic clubhouses were glued together around the base to form a closed polyhedron, it would have 80 faces. That is, the number of faces on an icosahedral geodesic dome with $n = 2$ i.e., n^2 or 4 faces on each of the 20 faces of an icosahedron. Note that it took 65 struts to form the geodesic clubhouse including the 10 struts for the base. If we stuck together two clubhouses to form a polyhedron, we would have 2×65 or 130 struts, but we would not need the extra 10 struts for the base. The new polyhedron

❸ c) cont’d

would have 120 struts or edges. Similarly, the icosahedral dome with 80 faces has $3 \times 80 \div 2$ or 120 edges. That is, the clubhouse and the icosahedral dome have the same number of faces and edges, and therefore the same number of vertices. The photo below shows two triangular faces of the icosahedron (in solid colored lines) and the four triangular subfaces (in dotted colored lines).

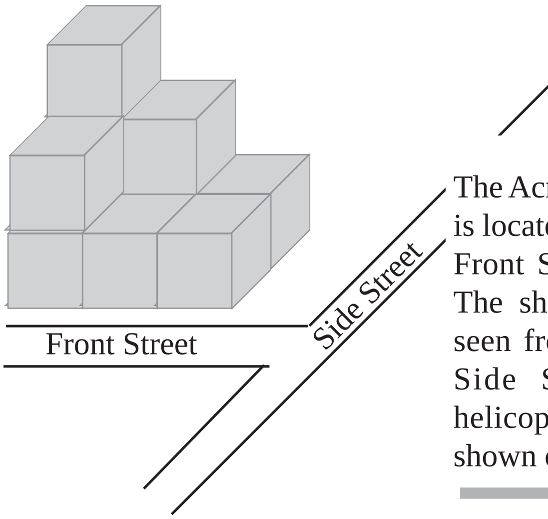




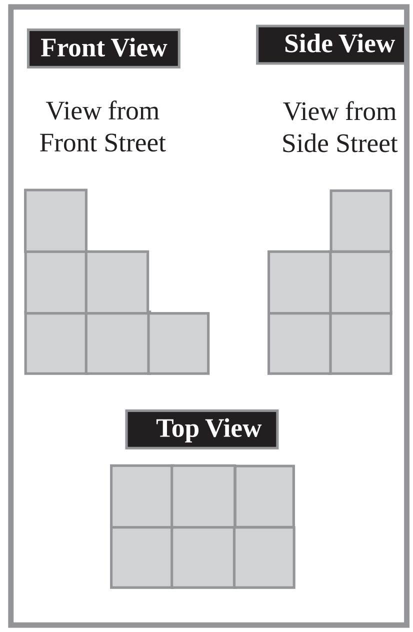
The icosahedral dome is shown in the same orientation as the geodesic clubhouse. This is an informal demonstration that the clubhouse is an icosahedral geodesic dome with $n = 2$.

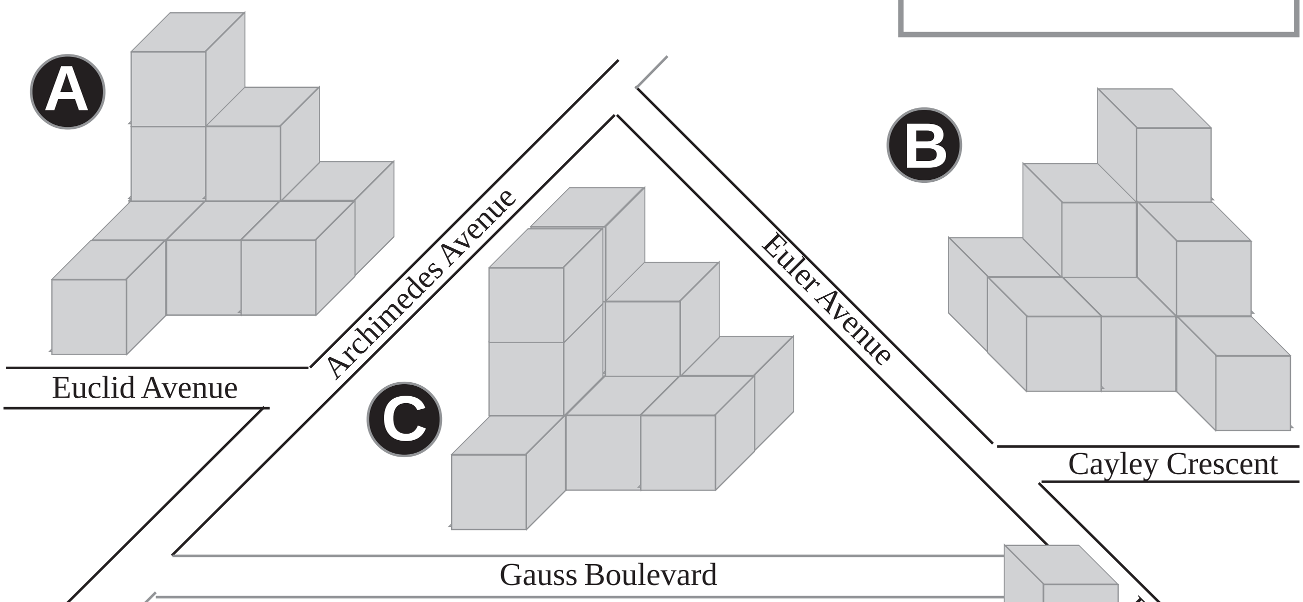
d) If Cinesphere were an icosahedral geodesic dome, then its triangles could be grouped into 20 congruent faces over its surface. Since we observe that the triangles get progressively smaller near the top, it does not have icosahedral symmetry.

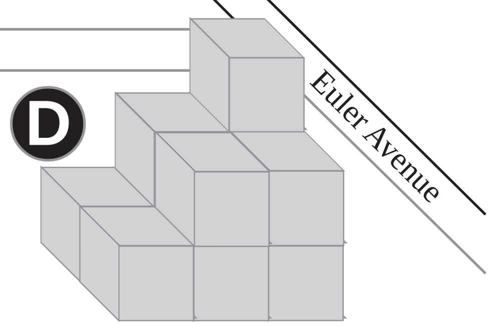
Spatial Visualization Warm-up



The Acme Microchip Building is located at the intersection of Front Street and Side Street. The shapes of this building as seen from Front Street, from Side Street, and from a helicopter directly above are shown on the right.



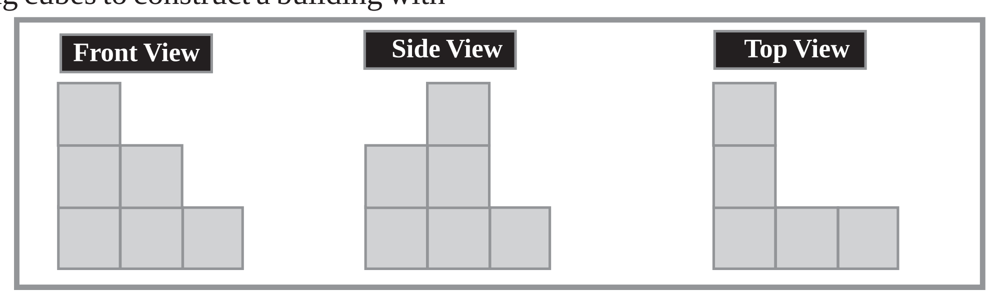




❶ Sketch diagrams on grid paper to show how each building would look from each of the streets on which it is located and from a helicopter directly above.

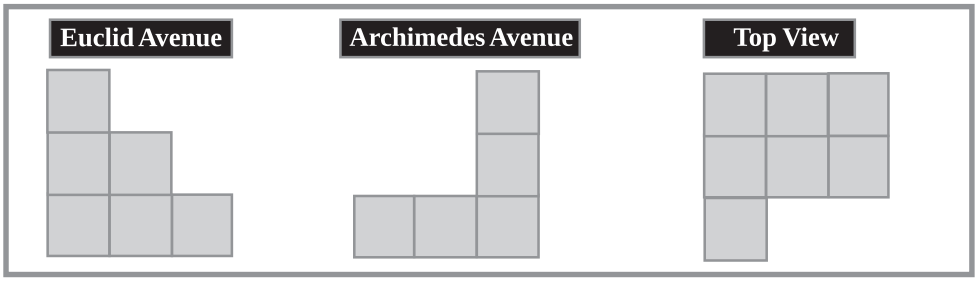
❷ Construct each building using interlocking cubes to check your sketches.

❸ Use interlocking cubes to construct a building with these views.

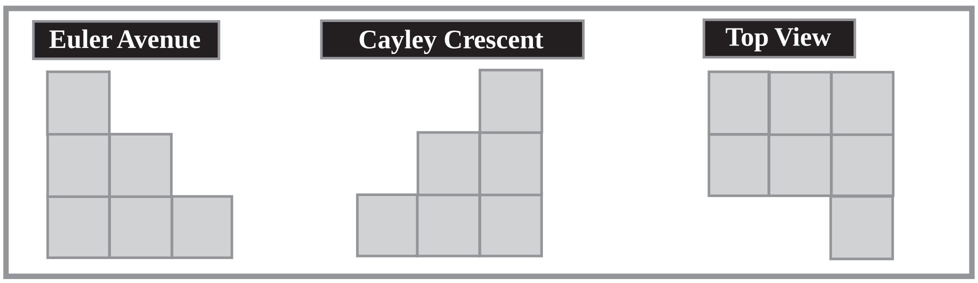


Answers to Spatial Visualization Warm-up

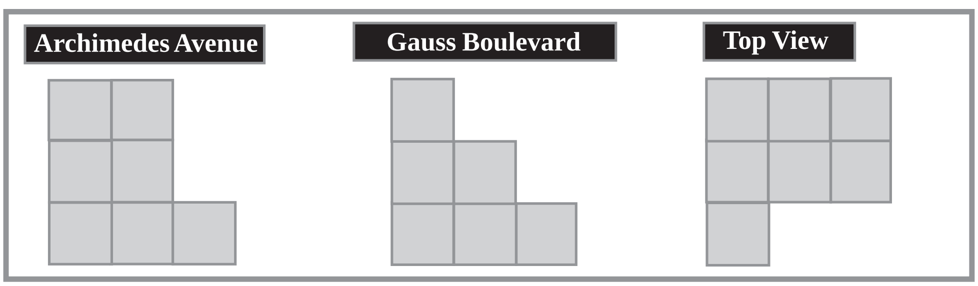
A



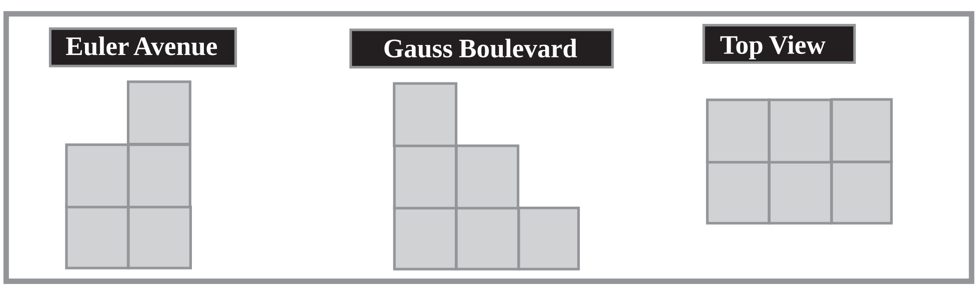
B



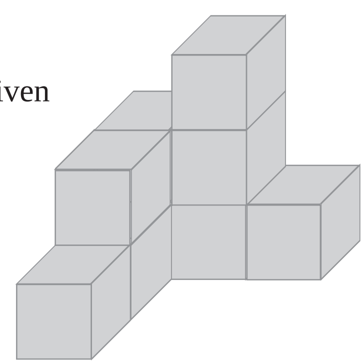
C



D



❸ One building with the given profiles looks like this.



TEMPLATE – GRID PAPER

impact math

focus on the future

PART IV

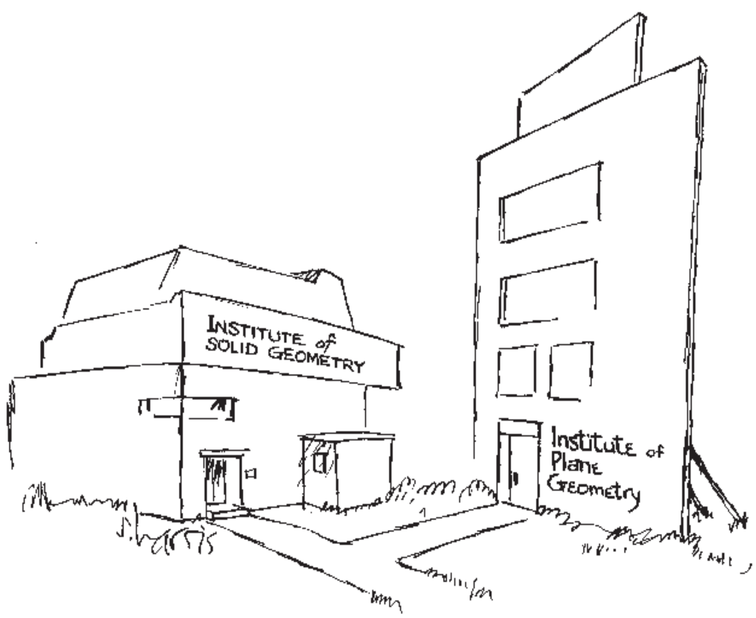
Geometry in Grade 8

Geometry and Spatial Sense: Grade 8

Overall Expectations

By the end of Grade 8, students will:

- identify, describe, compare, and classify geometric figures;
- identify, draw, and represent three-dimensional geometric figures;
- identify and investigate the relationships of angles;
- construct and solve problems involving lines and angles;
- investigate geometric mathematical theories to solve problems;
- use mathematical language effectively to describe geometric concepts, reasoning, and investigations.



Specific Expectations

(For convenient reference, the specific expectations are coded. G 8-1 refers to the first Geometry and Spatial Sense expectation in Grade 8.

Students will:

Three- and Two-Dimensional Geometry

G 8-1 - recognize three-dimensional figures from their top, side, and front views;

G 8-2 - sketch and build representations of three-dimensional figures (e.g., nets, skeletons) from front, top, and side views;

G 8-3 - identify the angle properties of intersecting, parallel, and perpendicular lines by direct measurement: interior, corresponding, opposite, alternate, supplementary, complementary;

G 8-4 - explore the relationship to each other of the internal angles in a triangle (they add up to 180˚) using a variety of methods (e.g., aligning corners of a paper triangle, using a protractor);

G 8-5 - investigate the Pythagorean relationship using area models and diagrams;

G 8-6 - solve angle measurement problems involving properties of intersecting line segments, parallel lines, and transversals;

G 8-7 - create and solve angle measurement problems for triangles;

G 8-8 - construct line segments and angles using a variety of methods (e.g., paper folding, ruler and compass);

G 8-9 - construct a circle given its centre and radius or centre and a point on the circle or three points on the circle;

G 8-10 - apply the Pythagorean relationship to numerical problems involving area and right triangles;

G 8-11 - describe the relationship between pairs of angles within parallel lines and transversals;

G 8-12 - explain why the sum of the angles of a triangle is 180˚;

G 8-13 - explain the Pythagorean relationship.

Activity 1 – Teacher Edition
What Is Shannon's Secret?

Expectations Addressed

This activity is dedicated to the overall expectation, "construct and solve problems involving lines and angles." Students will attempt to find by trial-and-error the triangle of minimal perimeter that can be inscribed in a given triangle on a grid. The specific expectations listed on page 59 are addressed in Activities 2, 3, and 4.

Context

The Geometry and Spatial Sense strand of the *Ontario Curriculum, Grades 1–8: Mathematics* contains a set of expectations in transformational geometry at each grade level from Grades 1 through 7. Typically, instructional materials develop these concepts as mathematical exercises with no apparent application. This sample unit attempts to help students discover how the properties of reflections can be applied in a powerful way to the solution of a *real world* optimization problem. It also guides students to discover how the Pythagorean Theorem can be applied to compute distances between two points on a grid and how some theorems about angles can yield angle measures without the need for measurement.

Before embarking on this unit, it will be important to check that students have mastered these concepts:

- how to compute the perimeter of a triangle
- how to use coordinates to represent points in the first quadrant
- how to apply the the Pythagorean theorem to compute the length of the hypotenuse of a right triangle
- how to obtain the image of a point under a reflection in a line by various methods, e.g., paper-folding, using a plastic mirror, etc.

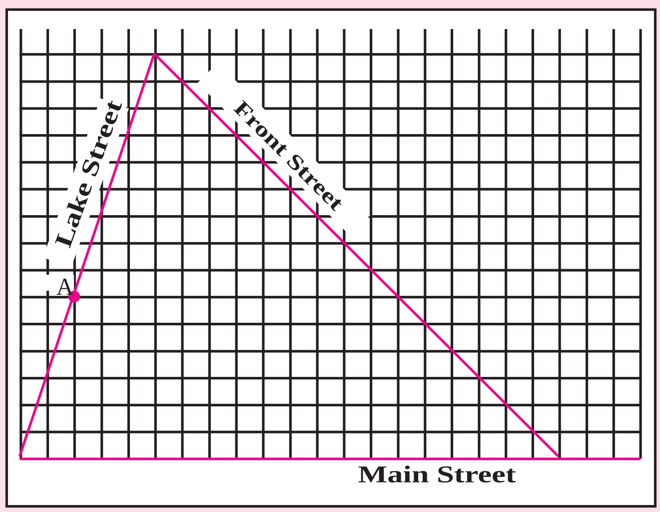
Activity 1 – Student Page

What Is Shannon's Secret?

Shannon is a well-known civil engineer. City planners consult with her when they need help solving problems in constructional design. Shannon's special techniques enable her to design structures that are efficient yet minimize wastage of materials. Few people understand her secret method, although they know it involves geometric thinking.

Currently Shannon is under contract by a world-class city to help them locate stations for their planned subway that will connect three major streets at minimum cost.

The grid below shows the three major streets: Main Street, Lake Street, and Front Street, and the location of the sports arena at point A. The city wants to find a grid point B on Front Street and a grid point C on Main Street so that the total distance AB + BC + CA is as small as possible. Shannon determined the locations of B and C in a few minutes. What is her secret?



What grid points B (on Front Street) and C (on Main Street) will Shannon choose as locations for the subway stations?

Activity 1 involves students in selecting a point B on Front Street and a point C on Main Street so that the perimeter of ΔABC, a subway connection network, is a minimum. Students do this by using a string on a geoboard, by employing trial-and-error on grid paper, or by using dynamic software such as *Sketchpad*. In Activity 2, students count squares on a grid and apply the Pythagorean theorem to calculate (rather than estimate) the perimeter of ΔABC for various choices of grid points B and C. Then in Activity 3, students discover how to find the optimum locations for B and C by reflecting in two different lines. (They verify inductively that this yields a solution that appears optimal, because a deductive proof is beyond most students at this level.) Finally, Activity 4 involves students in the application of the *sum of the angles in a triangle* theorem to determine the measures of the angles in ΔABC. All their findings are consolidated in a report to the City Planner indicating where subway stations B and C should be located.

Activity 1 – Teacher Edition

The Lesson Launch

5 minutes

Launch this lesson by distributing activity page 62. Have each student read the problem presented there. Then ask questions such as the following:

- *What is located at point A on Lake Street?*
- *How many different choices of location are possible for grid point B if it is to be located on Front Street between Lake Street and Main Street?*
- *How many different choices of location are possible for grid point C if it is to be located on Main Street between Lake Street and Front Street?*
- *What does the city want Shannon to do?*

Ensure that students understand Shannon's problem before proceeding to the next step. It is important that they realize that point A is fixed and that points B and C are to be chosen to minimize the perimeter of ΔABC.

Initiating Activity

10 minutes

Arrange students in pairs. Distribute geoboards (pegboards), elastic, and string to each pair. Have students hook the elastic around the pegs to represent the triangle formed by Lake Street, Front Street, and Main Street. Ask the students to locate the peg at point A on the triangle and loop one end of their string around this peg. Then have them loop the other end of the string around pegs on Front Street and Main Street and then back to point A, to form ΔABC. Have students measure the string to determine the perimeter of that triangle. Students should repeat this process until they think they have obtained a triangle of minimal perimeter. Then have students record the locations of these optimal positions for B and C.

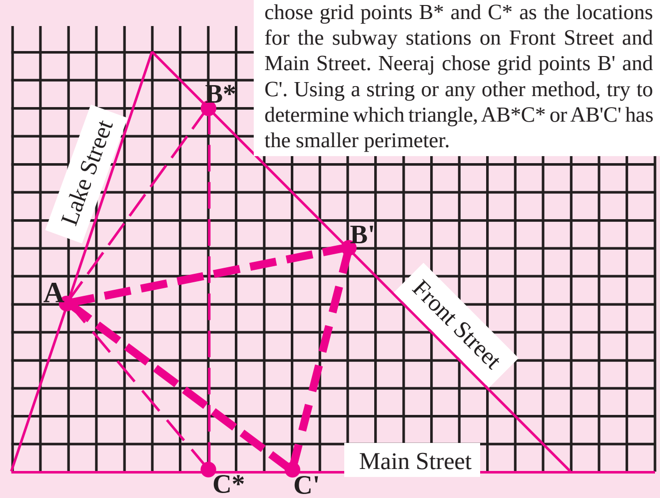
Paired Activity

25 minutes

Using the same pairs as in the Initiating Activity, provide students with a copy of page 63 and Template A (see p. 92). Have them complete Exercises ❶ – ❻ on page 63, and ensure that both students in each pair record the answers in their notebooks. [If you have any dynamical geometry software such as the *Geometer's Sketchpad* or *Cabri Geometry*, distribute the template *Using Sketchpad to Locate the Subway Stations* (see pp. 64-65) to students who complete the exercises ahead of the rest.]

Activity 1 – Student Page

The diagram below shows two different ways of choosing the grid points B and C. Andrea chose grid points B* and C* as the locations for the subway stations on Front Street and Main Street. Neeraj chose grid points B' and C'. Using a string or any other method, try to determine which triangle, AB*C* or AB'C' has the smaller perimeter.



Get a ruler marked in millimetres and the full-page copy of the grid above (Template A) from your teacher. (See page 92.)

❶ Measure the lengths of the line segments on your full-page grid to complete the missing lengths below. (where |AB*| means the distance from A to B*).

ANDREA'S TRIANGLE		NEERAJ'S TRIANGLE	
|AB*| =	_____cm	|AB'| =	_____cm
|B*C*| =	_____cm	|B'C'| =	_____cm
|C*A| =	_____cm	|C'A| =	_____cm

❷ Which choice of B and C, Andrea's or Neeraj's, yields a triangle of smaller perimeter? Explain how you decided.

❸ On your template, choose your own grid points B on Front Street and C on Main Street. With a pencil, join points A, B, and C to form a triangle ABC. Then measure and record these lengths in centimetres.

|AB| = ____ cm |BC| = ____ cm |CA| = ____ cm
perimeter ABC = _____cm.

❹ Repeat Exercise ❸ choosing different points B and C. Compare with your friends to see if any of them found a triangle with smaller perimeter.

❺ a) Which of your choices of B and C in Exercises ❷ and ❸ yielded the triangle of smallest perimeter?

b) Do you think points B and C can be chosen to yield a triangle with an even smaller perimeter than you or your classmates found? Explain why or why not.

❻ Explain why it is sometimes difficult to determine by measurement which of two triangles has the smaller perimeter.

Closure

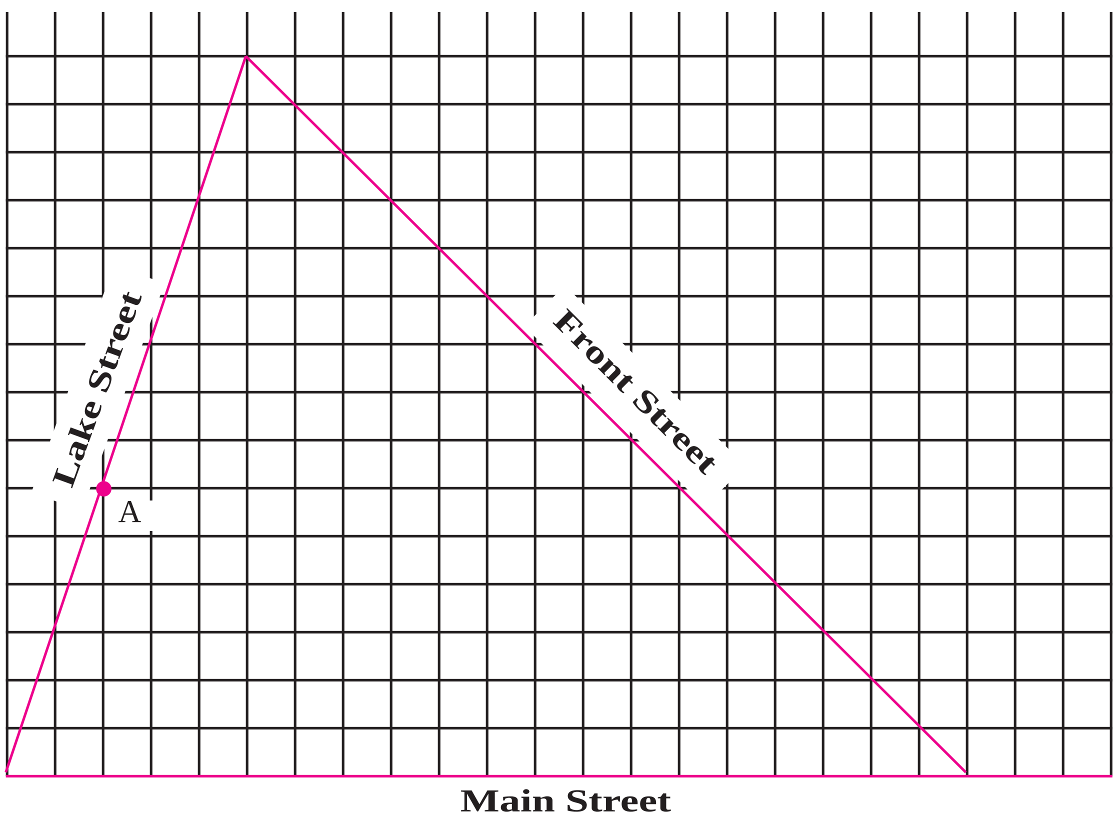
When all the students have finished, ask a member of each pair to record on the blackboard or overhead projector the perimeter of their optimal triangle (if less then 29.0 cm) and to sign their name beside it. Peruse the perimeters written on the blackboard to find the smallest perimeter listed there. (It should be 26.8 cm or larger.) When the lowest legitimate perimeter is determined, ask the students who obtained it to give the locations of points B and C. If the students have found the optimal locations for B(10, 10) and C(5, 0), ask how they obtained that solution. Praise the work of any pair who obtained this result. Ask students why trial-and-error is a difficult way to solve this problem. Explain that in Activity 3, they will discover Shannon's secret for locating points B and C without trial-and-error!

WHAT IS SHANNON'S SECRET?

Shannon is a well-known civil engineer. City planners consult with her when they need help solving problems in constructional design. Shannon's special techniques enable her to design structures that are efficient yet minimize wastage of materials. Few people understand her secret method, although they know it involves geometric thinking.

Currently Shannon is under contract by a world-class city to help them locate stations for their planned subway that will connect three major streets at minimum cost.

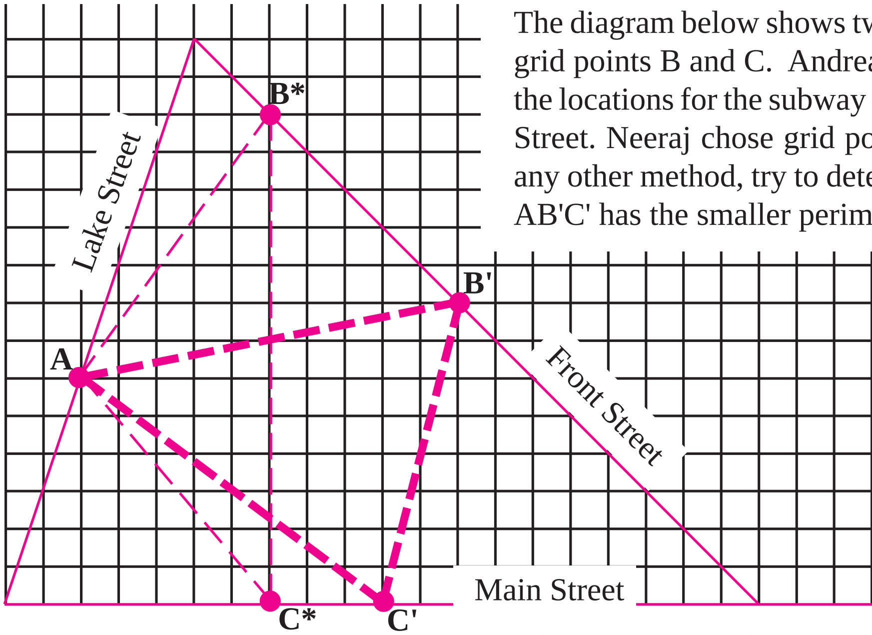
The grid below shows the three major streets: Main Street, Lake Street, and Front Street, and the location of the sports arena at point A. The city wants to find a grid point B on Front Street and a grid point C on Main Street so that the total distance AB + BC + CA is as small as possible. Shannon determined the locations of B and C in a few minutes. What is her secret?



What grid points B (on Front Street) and C (on Main Street) will Shannon choose as locations for the subway stations?

WHAT IS SHANNON'S SECRET?

The diagram below shows two different ways of choosing the grid points B and C. Andrea chose grid points B* and C* as the locations for the subway stations on Front Street and Main Street. Neeraj chose grid points B' and C'. Using a string or any other method, try to determine which triangle, AB*C* or AB'C' has the smaller perimeter.



Get a ruler marked in millimetres and the full-page copy of the grid above (Template A) from your teacher.

❶ Measure the lengths of the line segments on your full-page grid to complete the missing lengths below. (where |AB*| means the distance from A to B*).

ANDREA'S TRIANGLE	NEERAJ'S TRIANGLE
|AB*| = _____cm	|AB'| = _____cm
|B*C*| = _____cm	|B'C'| = _____cm
|C*A| = _____cm	|C'A| = _____cm

❷ Which choice of B and C, Andrea's or Neeraj's, yields a triangle of smaller perimeter? Explain how you decided.

❸ On your template, choose your own grid points B on Front Street and C on Main Street. With a pencil, join points A, B, and C to form a triangle ABC. Then measure and record these lengths in centimetres.

|AB| = ____ cm |BC| = ____ cm |CA| = ____ cm perimeter ABC = _____cm.

❹ Repeat Exercise ❸ choosing different points B and C. Compare with your friends to see if any of them found a triangle with smaller perimeter.

❺ a) Which of your choices of B and C in Exercises ❷ and ❸ yielded the triangle of smallest perimeter?

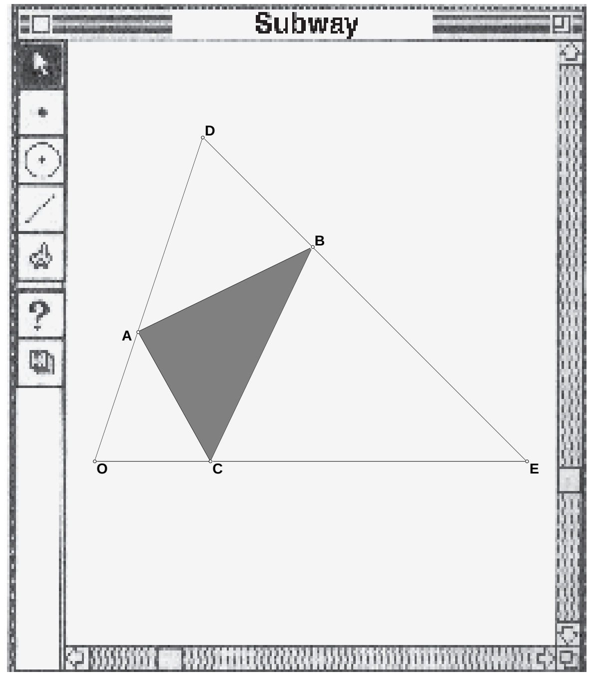
b) Do you think points B and C can be chosen to yield a triangle with an even smaller perimeter than you or your classmates found? Explain why or why not.

❻ Explain why it is sometimes difficult to determine by measurement which of two triangles has the smaller perimeter.

TEMPLATE

Using Sketchpad to Locate the Subway Stations

Sometimes you can use technology to model and solve problems. In this activity, you will locate points B and C to yield ΔABC of minimum perimeter using the *Geometer's Sketchpad*. Your investigation will take you through these steps.



❶ Construct ΔODE formed by Main Street, Lake Street, and Front Street.

❷ Create points A, B, and C on OD, (Lake Street), DE (Front Street), and OE (Main Street) to form ΔABC.

❸ Display the perimeter of ΔABC.

❹ Slide point B along DE (Front Street) to the position that yields the smallest perimeter.

❺ Slide point C along OE (Main Street) to the position that yields the smallest perimeter.

❻ Repeat steps ❹ and ❺ several times until a smaller perimeter cannot be found.

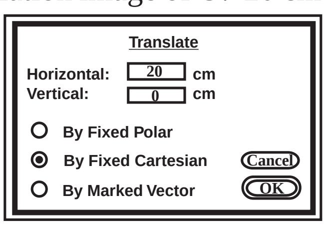
1 Construct ΔODE

Tool/Menu

Construct a point in the lower left corner of the screen. Label it O.

Select Point Tool and click
Display –> Show Label
Select Text Tool.
Double-click Label and enter O.

Construct the translation image of O: 20 cm right. Label it E.

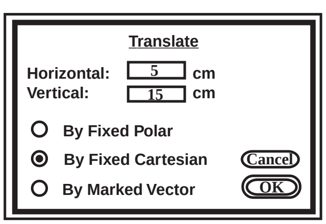


Select Point O.
Transform –> Translate…
Select *By Fixed Cartesian*
Enter 20 cm and 0 cm as shown in the dialogue box.
Select Text Tool.
Double-click Label and enter E.

Construct line segment OE.

Select O and E.
Construct –> Segment

Construct the translation image of O: 5 cm right, 15 cm up. Label it D.



Select Point O.
Transform –> Translate…
Select *By Fixed Cartesian*
Enter 5 cm and 15 cm in the dialogue box.
Select Text Tool.
Double-click Label and enter D.

Construct line segment OD.

Select O and D.
Construct –> Segment

Construct line segment DE.

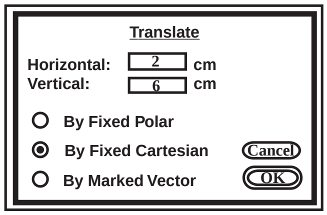
Select D and E.
Construct –> Segment

Using Sketchpad to Locate the Subway Stations

2 Construct ΔABC

Tool/Menu

Construct the translation image of O: 2 cm right, 6 cm up.
Label it A.



Select Point O.
Transform –> Translate…
Select *By Fixed Cartesian*
Enter 2 cm and 6 cm as shown in the dialogue box.
Select Text Tool.
Double-click Label and enter A.

Construct a point on segment DE.
Label it B.

Select line segment DE.
Construct –> Point On Object
Select Text Tool.
Double-click Label and enter B.

Construct a point on segment OE.
Label it C.

Select line segment OE.
Construct –> Point On Object
Select Text Tool.
Double-click Label and enter C.

Construct line segment AB.

Select A and B.
Construct –> Segment

Construct line segment BC.

Select B and C.
Construct –> Segment

Construct line segment BC.

Select C and A.
Construct –> Segment

3 Display Perimeter ΔABC

Select points A, B, and C.
Construct the interior of ΔABC.
Select *Perimeter* from the *Measure* menu.

Select points A, B, and C.
Construct –> Polygon Interior
Measure –> Perimeter

4 Slide B to Minimize Perimeter.

Drag point B slowly along DE until the perimeter reaches its lowest value.

Click and drag point B.

5 Slide C to Minimize Perimeter.

Drag point C slowly along OE until the perimeter reaches its lowest value.

Click and drag point C.

6 Repeat Steps 4 & 5

Repeat steps 4 and 5 several times until a smaller perimeter cannot be found. Record this perimeter.

Grade 8

Answer Key for Activity 1

❶ The lengths of the line segments are given below.

ANDREA'S TRIANGLE	NEERAJ'S TRIANGLE
|AB*| = 8.6 cm	|AB'| = 10.2 cm
|B*C*| = 13.0 cm	|B'C'| = 8.2 cm
|C*A| = 7.8 cm	|C'A| = 10.0 cm

❷ Andrea's triangle has perimeter 29.4 cm.
Neeraj's triangle has perimeter 28.4 cm.
Neeraj's choice of B and C yields the triangle of smaller perimeter.

❸ Answers will vary, although we may expect the perimeter of ΔABC to be smaller than the perimeter of Andrea's triangle, i.e., 29.4 cm.

❹ Usually students find someone who has created a triangle of smaller perimeter than the one they constructed.

❺ a) Answers will vary.
b) Students have no way of knowing whether their answers are optimal, but when they find other students with a triangle of equal or smaller perimeter than theirs, they tend to think this triangle is optimal. The optimal triangle has vertices B and C located on the grid points (10, 10) and (5, 0) respectively, as shown below.

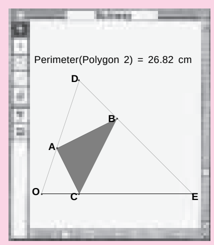
It has an optimal perimeter of $12\sqrt{5}$ or 26.8 cm.

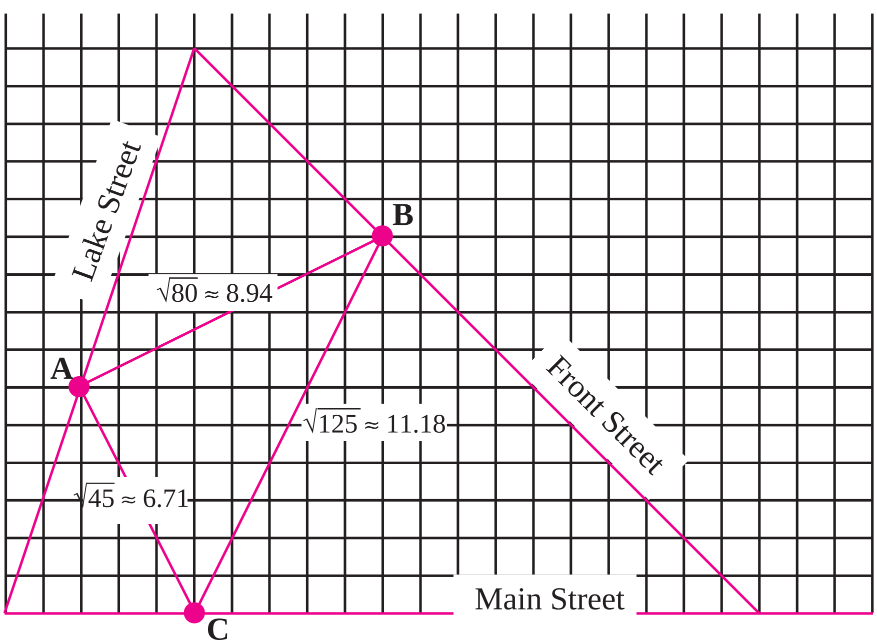
❻ Students may discover that the perimeters of two different triangles can be so close in value, that their difference is inside the range of measurement error. In such a case it is difficult to determine by measurement which triangle has the greater perimeter. Recognition of this fact leads naturally to the search for a method for calculating the lengths of the sides of a triangle from the coordinates of its vertices. This is the focus of Activity 2.

Teacher Note

Triangle ABC, shown on the grid below, has the smallest perimeter of all triangles with vertex A(2, 6) and vertices B and C on Front Street and Main Street respectively. The requirement that B and C be grid points is not necessary, because the process used to obtain ΔABC (see Activity 3) yields the triangle of minimum perimeter whether or not B and C are grid points!

Students who use Sketchpad will be able to attain a minimal perimeter close to 26.8, but there may be a discrepancy of ±0.1 because of the pixel limits.





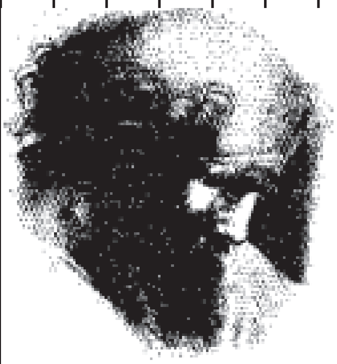
Teacher Note

Applying the Pythagorean Theorem, we calculate the sides of ΔABC to have the lengths shown in the diagram. If we write these radicals in simplest form, we obtain:

$|AC| = \sqrt{45} = 3\sqrt{5}$
$|AB| = \sqrt{80} = 4\sqrt{5}$
$|BC| = \sqrt{125} = 5\sqrt{5}$

That is, the sides of ΔABC are in the ratio 3:4:5, indicating that it is a right triangle. In Activity 4, the students will discover that ΔABC is a right triangle by using the converse of the Pythagorean Theorem.

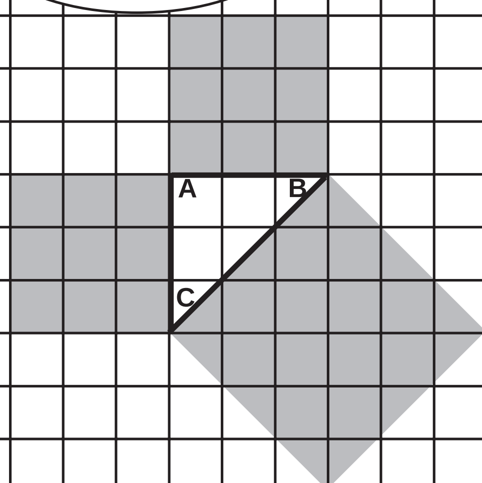
The Pythagorean Relationship



Pythagoras
582 B.C – 501 B.C.

The postage stamp shown here celebrates the Pythagorean relationship that was discovered over 2500 years ago. It shows a right triangle with a square drawn on each side of the triangle. Each small square has an area of one square unit. Count the small squares to record the area of the square on each side. Describe what you discover. This is true for all right triangles and is called the *Pythagorean relationship* or *Pythagorean theorem*. Named in honor of the mathematician, Pythagoras, of ancient Greece, it remains the most famous theorem in mathematics.

Count squares and half-squares to show that the Pythagorean relationship is true for ΔABC below.



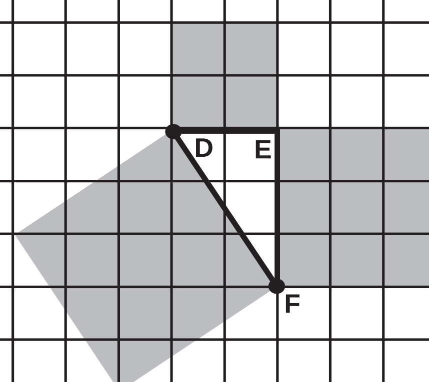
Use the area of the square on side BC of ΔABC to calculate the length of side BC.

Explain how you can use the lengths of the two shorter sides of a right triangle to calculate the length of the longest side (called the *hypotenuse*).

Use the Pythagorean relationship to determine:

- the length of side DF.
- the distance between points D and F.

Explain how you can calculate the distance between two points on a grid by counting squares.



Activity 2 – Teacher Edition

Using Coordinates to Compute Distances

Expectations Addressed

G 8-5 investigate the Pythagorean relationship using area models and diagrams.

G 8-10 apply the Pythagorean relationship to numerical problems involving area and right triangles.

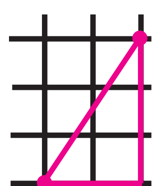
G 8-13 explain the Pythagorean relationship.

Context

It is unlikely that you are familiar with units such as *chains* described in the article presented here. Your students may be equally unfamiliar with *acres*, *feet*, and *miles*. However, they may be interested to learn that the units that they have worked with through their school years may be different from those used by their parents or grandparents. As the article points out, the fact that the concessions of Ontario and Quebec were laid out on a grid with grid lines spaced 2 km apart was a lucky *accident*!
[1 mile ≈ 1.609 km; so $1^{1}/_{4}$ miles ≈ 2.011 km.]

Calculating Distances by Counting Squares

Prior to this activity, students should have discovered how to use the Pythagorean relationship to calculate the distance between any two points on a grid. A template, *The Pythagorean Relationship* (see p. 67), is provided as a bridge into this activity for students lacking the necessary background. It is NOT intended that students be given the distance formula. Rather it is expected that students will count horizontally and vertically the number of squares between two grid points and use the Pythagorean relationship to calculate the distance between them. That is, students must understand the distance between two grid points as the length of the hypotenuse of the right triangle shown here.



Calculating Distances by Using Coordinates

Once students are able to calculate the distance between two grid points by counting squares, they are ready to discover how to calculate the distance between two grid points by examining the coordinates of these points. (Recall that the Geometry and Spatial Sense strand at the Grade 6 level calls for "an understanding of coordinates in a Cartesian plane in the first quadrant.") By subtracting the coordinates of two grid points, students can determine the number of squares horizontally and vertically between two grid points. Then they can apply the method described above to calculate the distance between them.

Activity 2 – Student Page

Using Coordinates to Compute Distances

C2 THE TORONTO STAR Thursday, July 16, 1998

YOU ASKED US

WALTER STEFANIUK

Toronto's grid in kilometres

Q Why are Toronto's major streets exactly two kilometres apart since they were laid out long before we adopted the metric system?

A It's a coincidence that Queen St., Bloor St., St. Clair Ave., Eglinton Ave., and north onwards into the coun tryside happen to be almost two kilometres apart. It's the same with the major streets running east to west.

The land was surveyed in grids —measuring 100 chains. The surveyor's chain is 66 feet long, and 100 chains by 100 chains covers 1,000 acres. This provided for easy subdivision of the land in the early years of North American settlement and concession roads across Ontario and Quebec were built along the grids.

Now, 100 surveyor chains equals 6,600 feet, which is $1^{1}/4$ miles. So the concession lines that became Queen, Bloor, St. Clair, Eglinton, etc. were laid out $1^{1}/4$ miles apart. And two kilometres equals 1.24 miles.

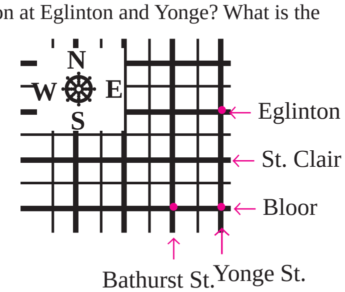
The fact that $1^{1}/4$ miles turns out to be just over two kilometres is purely chance.

❶ What is a grid, and why do you think a city would be laid out in this way?

❷ a) What is the distance between any two adjacent major streets (shown by a thick line) that run parallel?

b) What is the distance between the subway station at Bloor and Yonge and the station at Eglinton and Yonge? What is the distance between the subway station at Bloor and Bathurst and the station at Bloor and Yonge?

❸ How long a track would be needed to connect directly the stations at Bloor and Bathurst and at Eglinton and Yonge?



Activity 2 – Teacher Edition

The Lesson Launch 5 minutes

Prior to launch, ensure that students can apply the Pythagorean theorem to determine lengths of line segments (see p. 67). To launch the lesson, distribute student page 70. Have the students read the newspaper article and then pose questions such as:

- *What is meant by a "grid"?*
- *What is meant by the statement "the distance between Elm Street and Oak Street is 5 km"?*
- *When is the distance between two streets zero?*
- *What do we know about two streets on a grid if the distance between them is greater than zero?*

Ensure that students understand that a grid is a network of horizontal and vertical equally-spaced lines. They should also understand that the distance between two streets has meaning only if the streets are parallel.

Initiating Activity 10 minutes

Ask students to work individually on Exercises ❶, ❷, and ❸ and record their answers in their notebooks. When the students have finished, discuss the answers to these exercises, ensuring that all students understand how to count squares on the grid and apply the Pythagorean relationship to calculate the distance between any two grid points.

Paired Activity 10 minutes

Group the students in the same pairs as in Activity 1. Have each pair complete Exercises ❹ and ❺ and compare their answers with the results they obtained in Activity 1. When students have finished this activity, discuss with the class the advantages and disadvantages of calculating vs. measuring to find the perimeter of various triangles.

Individual Activity 10 minutes

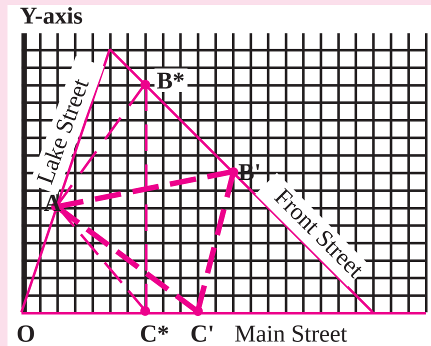
Ask students questions such as the following:

What are the coordinates of points A and B on the grid?
What is the horizontal distance between A and B?
What is the vertical distance between A and B?
How can we use the horizontal and vertical distances to find the distance between A and B?

Once it is clear that most students understand how to calculate the distance between two grid points by using the Pythagorean relationship, assign them Exercises ❻ and ❼ to work on individually.

Activity 2 – Student Page

Using Coordinates to Compute Distances



The city for which Shannon is choosing the subway stations is laid out on a grid with a distance of 1 km between parallel streets.

❹ To calculate the distance from O to B*, draw a right triangle that has hypotenuse OB*. Count squares to find the lengths of the other two sides of your right triangle. Then use the Pythagorean relationship to calculate the length of OB*.

Most North American cities are laid out on a grid. This enables us to describe the location of any grid point by a pair of integer coordinates. The first coordinate gives the (horizontal) distance of the intersection from the Y-axis and the second coordinate gives its (vertical) distance from the X-axis (shown as Main Street). The intersection of these axes is the origin O(0,0). By observing the difference in the corresponding coordinates of two grid points, we can determine the horizontal and vertical distances between them. This enables us to calculate the distance between them.

❺ Write the coordinates of points A, B', C', B*, and C*. Use the Pythagorean relationship to calculate the following distances.
(where AB* means the distance from A to B*).

Andrea's Triangle		Neeraj's Triangle	
|AB*| =	_____km	|AB'| =	_____km
|B*C*| =	_____km	|B'C'| =	_____km
|C*A| =	_____km	|C'A| =	_____km

Calculate the perimeters of ΔAB*C* and ΔAB'C'. Compare your answers above with the answers you obtained to Exercises ❶ and ❷ in Activity 1. Which method do you prefer – measuring or calculating? Explain why.

❻ Record the coordinates of points B and C you found in Exercise ❺ a of Activity 1 and use these coordinates to calculate the lengths of the sides of ΔABC. Then calculate and record the perimeter of ΔABC.

A has coordinates (2, 6)
B has coordinates (_ , _)
C has coordinates (_ , _)
|AB| = _____ km
|BC| = _____ km
|CA| = _____ km
Perimeter ΔABC = ____km

❼ Were you able to choose locations B and C that yielded a triangle of smaller perimeter than Andrea's or Neeraj? Do you think that trial-and-error is the best way to find the locations of B and C that yield the triangle of smallest perimeter? Explain why or why not.

Closure

Discuss the students' answers to Exercises ❻ and ❼. Consolidate their understanding of the process for calculating the distance between the two grid points A and B' by drawing the right triangle with hypotenuse AB'. Show how the length of the hypotenuse can be calculated by counting squares and how this is the same as subtracting coordinates. Reinforce with the students the observation that a trial-and-error method for finding an optimum solution is not very practical when there are a large number of possible cases.

Using Coordinates to Compute Distances

C2 THE TORONTO STAR Thursday, July 16, 1998

YOU ASKED US

WALTER STEFANIUK

Toronto's grid in kilometres

Q Why are Toronto's major streets exactly two kilometres apart since they were laid out long before we adopted the metric system?

A It's a coincidence that Queen St., Bloor St., St. Clair Ave., Eglinton Ave., and north onwards into the countryside happen to be almost two kilometres apart. It's the same with the major streets running east to west.

The land was surveyed in grids —measuring 100 chains. The surveyor's chain is 66 feet long, and 100 chains by 100 chains covers 1,000 acres. This provided for easy subdivision of the land in the early years of North American settlement and concession roads across Ontario and Quebec were built along the grids.

Now, 100 surveyor chains equals 6,600 feet, which is $1^1/4$ miles. So the concession lines that became Queen, Bloor, St. Clair, Eglinton, etc. were laid out $1^1/4$ miles apart. And two kilometres equals 1.24 miles.

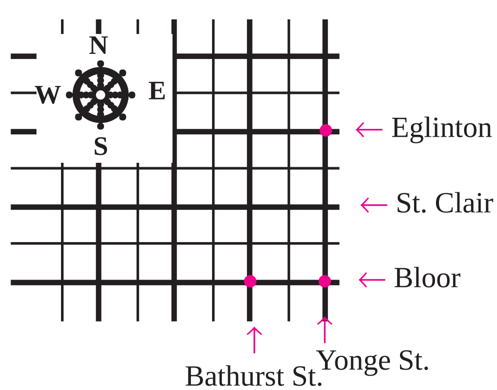
The fact that $1^1/4$ miles turns out to be just over two kilometres is purely chance.

❶ What is a grid, and why do you think a city would be laid out in this way?

❷ a) What is the distance between any two adjacent major streets (shown by a thick line) that run parallel?

b) What is the distance between the subway station at Bloor and Yonge and the station at Eglinton and Yonge? What is the distance between the subway station at Bloor and Bathurst and the station at Bloor and Yonge?

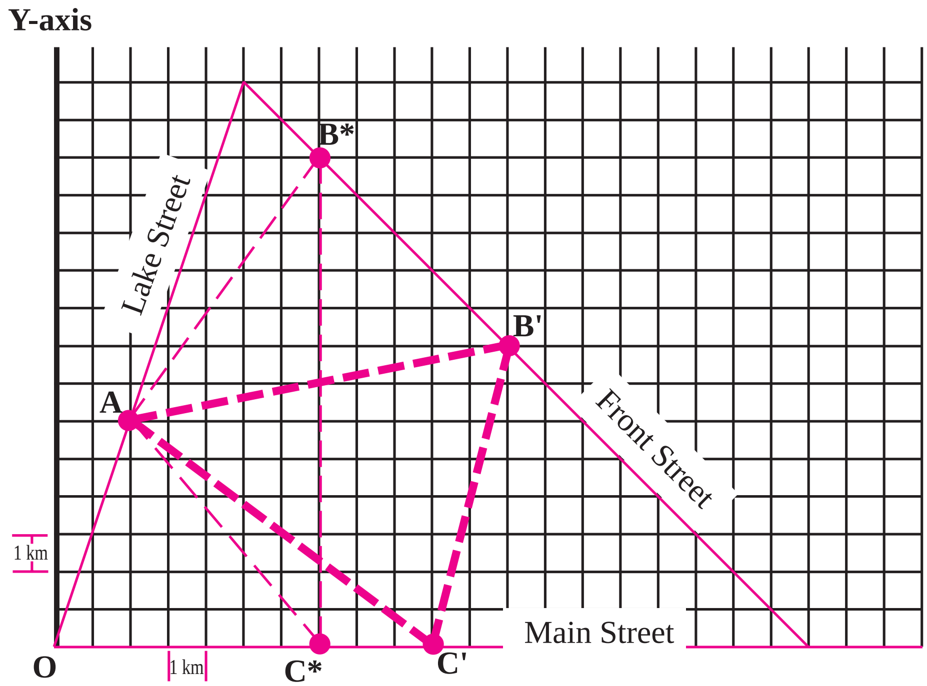
❸ How long a track would be needed to connect directly the stations at Bloor and Bathurst and at Eglinton and Yonge?



USING COORDINATES TO COMPUTE DISTANCES

The city for which Shannon is choosing the subway stations is laid out on a grid with a distance of 1 km between parallel streets.

❹ To calculate the distance from O to B*, draw a right triangle that has hypotenuse OB*. Count squares to find the lengths of the other two sides of your right triangle. Then use the Pythagorean relationship to calculate the length of OB*.



Most North American cities are laid out on a grid. This enables us to describe the location of any grid point by a pair of integer coordinates. The first coordinate gives the (horizontal) distance of the intersection from the Y-axis and the second coordinate gives its (vertical) distance from the X-axis (shown as Main Street). The intersection of these axes is the origin O(0,0). By observing the difference in the corresponding coordinates of two grid points, we can determine the horizontal and vertical distances between them. This enables us to calculate the distance between them.

❺ Write the coordinates of points A, B', C', B*, and C*. Use the Pythagorean relationship to calculate the following distances.
(where |AB*| means the distance from A to B*).

ANDREA'S TRIANGLE				NEERAJ'S TRIANGLE			
|AB*|	=	_____	km	|AB'|	=	_____	km
|B*C*|	=	_____	km	|B'C'|	=	_____	km
|C*A|	=	_____	km	|C'A|	=	_____	km

Calculate the perimeters of ΔAB*C* and ΔAB'C'. Compare your answers above with the answers you obtained to Exercises ❶ and ❷ in Activity 1. Which method do you prefer – measuring or calculating? Explain why.

❻ Record the coordinates of points B and C you found in Exercise ❺ a of Activity 1 and use these coordinates to calculate the lengths of the sides of ΔABC. Then calculate and record the perimeter of ΔABC.

A has coordinates (2, 6)
B has coordinates (_ , _)
C has coordinates (_ , _)
|AB| = _____ km
|BC| = _____ km
|CA| = _____ km
Perimeter ΔABC = ____km

❼ Were you able to choose locations B and C that yielded a triangle of smaller perimeter than Andrea's or Neeraj? Do you think that trial-and-error is the best way to find the locations of B and C that yield the triangle of smallest perimeter? Explain why or why not.

GRADE 8

ANSWER KEY FOR ACTIVITY 2

❶ A grid is an array of squares formed by equally spaced horizontal and vertical lines. It is used to locate points relative to these lines.

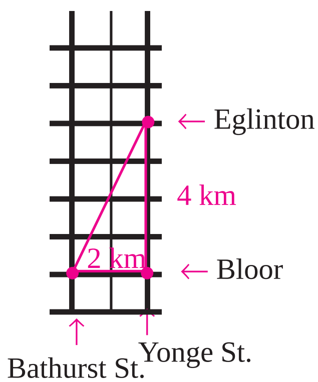
When cities are laid out on a grid, it is easy to identify a location by referring to its distance from a particular intersection (or grid point).

❷ a) The distance between two adjacent major streets on the grid is about 2 km.
b) The distance between the subway station at Bloor and Yonge and the station at Eglinton and Yonge is about 4 km.

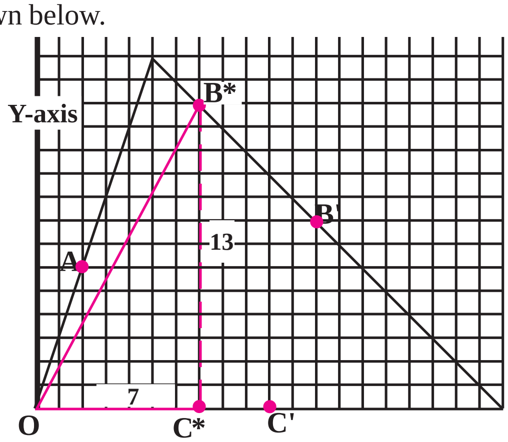
The distance between the subway stations at Bloor and Bathurst and Bloor and Yonge is about 2 km.

❸ The distance between the subway station at Bloor and Bathurst and the station at Eglinton and Yonge is the length of the hypotenuse of the right triangle shown below.

The length of the hypotenuse is the square root of the sum $4^2 + 2^2$, that is, $\sqrt{20}$ or 4.47 km. Therefore the length of track needed would be about 4.5 km.

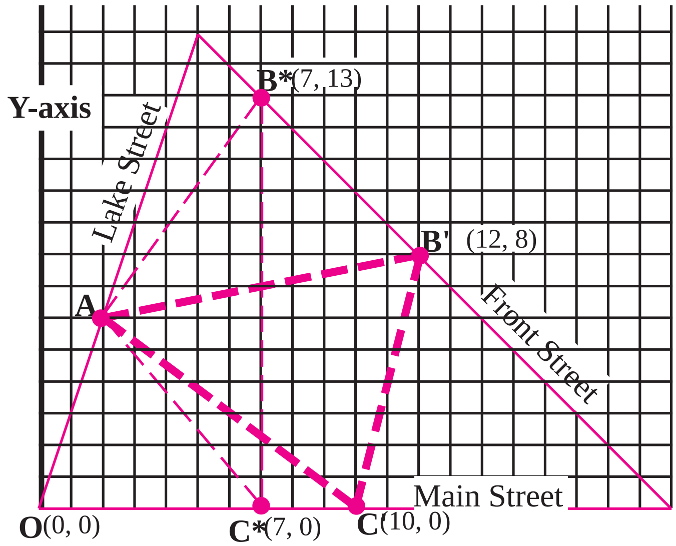


❹ The right triangle of which OB* is the hypotenuse is shown below.



Then $|OB^*| = \sqrt{7^2 + 13^2}$, or $\sqrt{218} \approx 14.76$.

❺ The coordinates given below show the horizontal and vertical distances.



To obtain |AB*| it is expected that students will count horizontally 5 spaces and vertically 7 spaces from A to B*. Then they should apply the Pythagorean relationship to obtain $|AB^*| = \sqrt{5^2 + 7^2}$, or $\sqrt{74}$. In this way, they should obtain the following distances.

ANDREA'S TRIANGLE	NEERAJ'S TRIANGLE
$\lvert AB^*\rvert = \sqrt{74}$ or 8.6 km	$\lvert AB'\rvert = \sqrt{104}$ or 10.2 km
$\lvert B^*C^*\rvert =$ 13 km	$\lvert B'C'\rvert = \sqrt{68}$ or 8.2 km
$\lvert C^*A\rvert = \sqrt{61}$ or 7.8 km	$\lvert C'A\rvert = \sqrt{100}$ or 10.0 km

The perimeters of ΔAB*C* and ΔAB'C' are respectively 29.4 km and 28.4 km. These are the same values given in Exercise ❷ of Activity 1, p. 63.

❻ Answers will vary depending on the choice of points B and C. As noted in the answer to Exercise ❺ p. 64, students have no way of knowing whether their answers are optimal. The optimal triangle has vertices B and C located on the grid points (10, 10) and (5, 0) respectively. It has an optimal perimeter of $12\sqrt{5}$ or 26.8 km, so we can expect most students' perimeters to be a little higher than this number.

❼ Students may discover that finding the triangle of smallest perimeter by trial-and-error is not efficient, because there are so many possible choices of vertices B and C.

The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 2

APPLICATION	Level 1	Level 2	Level 3	Level 4
Apply the Pythagorean Relationship to Estimate & Calculate Distances on a Grid (exercises ❹ – ❼) G 8-5, G 8-10	• There are several errors in the coordinates of the points on the grid. • The Pythagorean relationship is not used to calculate the distances between grid points and/or lengths of line segments.	• The coordinates of the points on the grid are given with little or no errors. • The Pythagorean relationship is used to calculate the distances between grid points and lengths of line segments but there are several errors.	• The coordinates of the points on the grid are given with little or no errors. • The Pythagorean relationship is used to calculate the distances between grid points and lengths of line segments with little or no errors.	In addition to Level 3, • The perimeters of Andrea's and Neeraj's triangles are calculated correctly. • There is evidence that the student understands that calculating distances on a grid is more accurate than measuring distances. • The student uses estimation to find a triangle of perimeter less than a given triangle.

Achievement Levels Defined by the Ministry of Education and Training

Level 1 Identifies achievement that falls much below the provincial standard.

Level 2 Identifies achievement that approaches the standard.

Level 3 The "provincial standard," identifies a high level of achievement of provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.

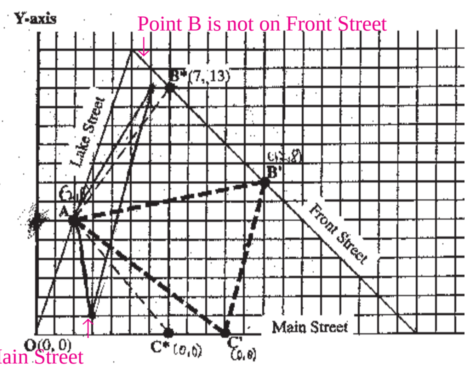
Level 4 Identifies achievement that surpasses the standard.

WHAT YOU MIGHT SEE

Level 2

The city for which Shannon is choosing the subway stations is laid out on a grid with a distance of 1 km between parallel streets. Main Street is the X-axis. The Y-axis is the vertical street marked on the grid. The intersection of these axes is shown as the origin O(0,0).

❹. On this grid mark the coordinates of the locations A, B', C*, and C'. Observe that the coordinates of B* are marked for you.



❺. Using the same method you applied to the Toronto grid, calculate these distances. (where AB* means the distance from A to B*).

ANDREA'S TRIANGLE	NEERAJ'S TRIANGLE
AB* = 4.5 km	AB' = 5.3 km
B*C* = 6.6 km	B'C' = 4.4 km
C*A = 4.1 km	C'A = 5.2 km

Calculate the perimeters of ΔAB*C* and ΔAB'C'. Compare your answers above with the answers you obtained to Exercises ❶ and ❷ in Activity 1. Which method do you prefer – measuring or calculating? Explain why.

❻. Choose intersections B and C on your grid to obtain ΔABC with as small a perimeter as you can. Record the coordinates of B and C and calculate the lengths of the sides of ΔABC. Then calculate and record the perimeter of ΔABC.

A has coordinates (0, 6)
B has coordinates (6, 13)
C has coordinates (3, 1)
AB = 4.2 km
BC = 6.5 km
CA = 2.7 km
Perimeter ΔABC = 13.5 km

❼. Were you able to choose locations B and C that yielded a triangle of smaller perimeter than Andrea's or Neeraj's? Do you think that trial-and-error is the best way to find the locations of B and C that yield the triangle of smallest perimeter? Explain why or why not.

Note: These samples of student work were taken from an earlier draft of Activity 2, so the wording is slightly different from that shown on page 71.

This student wrote his answers on the question sheet as shown. The coordinates of the points on the grid were given without error. However, the student was unable to apply the Pythagorean relationship to obtain the correct lengths of line segments. The perimeters of Andrea and Neeraj's triangles (submitted on a separate sheet) were incorrect because the side lengths of the triangles were incorrect. The student knew that the perimeter of a triangle is obtained by adding the side lengths but was unable to find locations for B and C that would yield a triangle of smaller perimeter than those of Andrea and Neeraj because he failed to understand that these points had to be located on Front and Main Streets. This student requires some instruction on how to apply the Pythagorean relationship.

Level 3

❺. Using the same method you applied to the Toronto grid, calculate these distances. (where AB* means the distance from A to B*).

$6^2 = 36$
$5^2 = 25$
61

ANDREA'S TRIANGLE	NEERAJ'S TRIANGLE
AB* = 8.602325 km	AB' = 10.44030303 km
B*C* = 13 km	B'C' = 8.246211251 km
C*A = 7.81102496 km	C'A = 10 km

Calculate the perimeters of ΔAB*C* and ΔAB'C'. Compare your answers above with the answers you obtained to Exercises ❶ and ❷ in Activity 1. Which method do you prefer – measuring or calculating? Explain why.

B*C* = 29.41257494 ΔAB'C' = 28.44425028 I perfer calculating if I need to get a very accurate results, but measuring is much faster, but I perfer to calculat to get my results to get more accurate result.

❻. Choose intersections B and C on your grid to obtain ΔABC with as small a perimeter as you can. Record the coordinates of B and C and calculate the lengths of the sides of ΔABC. Then calculate and record the perimeter of ΔABC.

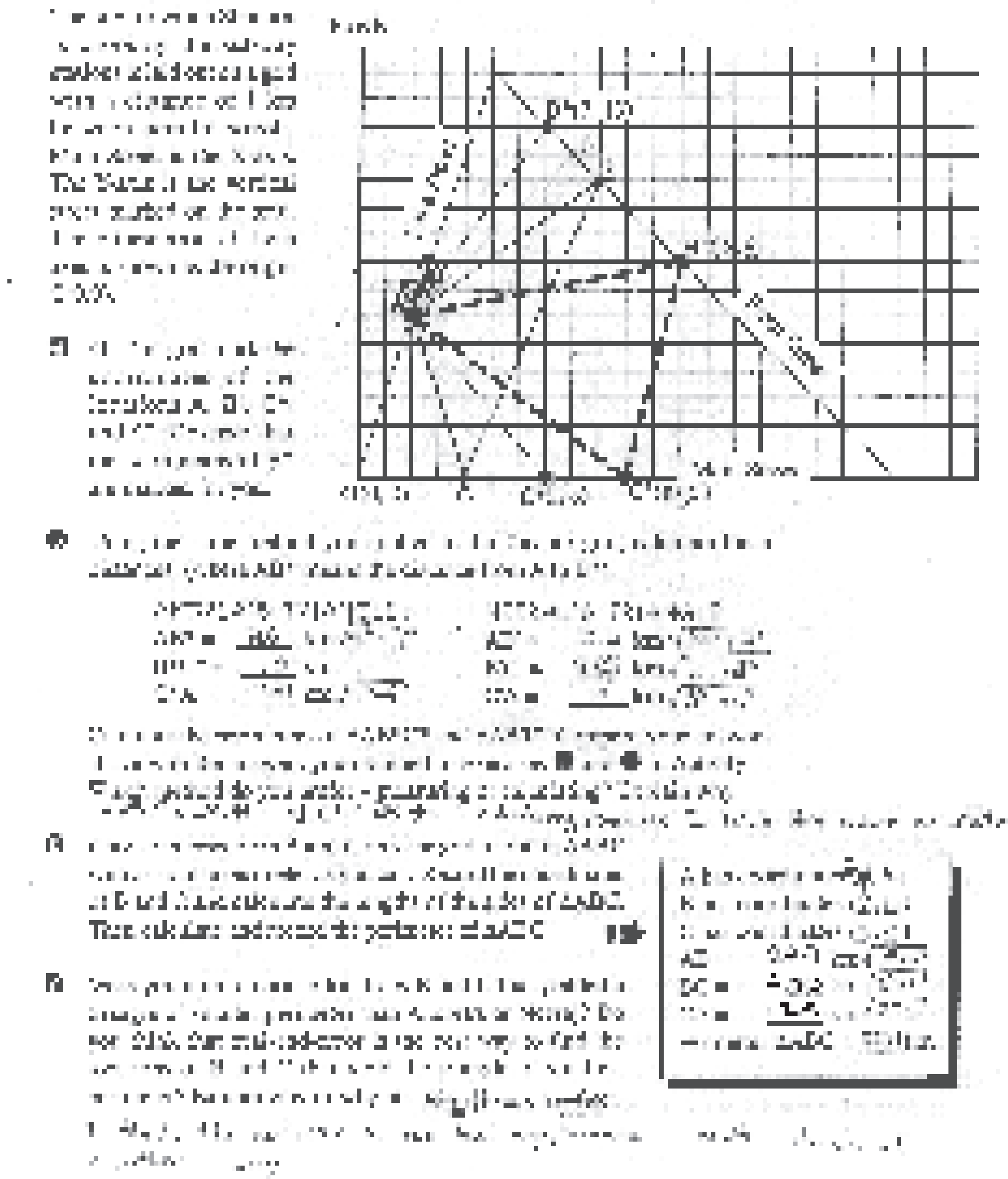
A has coordinates (0, 6)
B has coordinates (8, 12)
C has coordinates (0, 0)
AB = 4.472136 km
BC = 10 km
CA = 6.32455532 km
Perimeter ΔABC = 20.796691 km
12.8

❼. Were you able to choose locations B and C that yielded a triangle of smaller perimeter than Andrea's or Neeraj? Do you think that trial-and-error is the best way to find the locations of B and C that yield the triangle of smallest perimeter? Explain why or why not. No, on the orange sheet I didn't get a smaller triangle but I do think that trial-&-error is the best way to try & get a smaller perimoter becouse you make a triangle & see what.

This student wrote her answers on the question sheet. The coordinates of the points on the grid were given without error. The student also applied the Pythagorean relationship to obtain the correct lengths of line segments in all cases except in Exercise ❻. The answers were given to as many decimal places as the calculator would display, but this is not an error since the level of precision was not specified. The perimeters of Andrea and Neeraj's triangles were calculated correctly. This student was unable to estimate distances accurately enough to find locations for B and C that would yield a triangle of smaller perimeter than those of Andrea and Neeraj. However, the student understands that calculating distances on a grid is more accurate than measuring distances.

WHAT YOU MIGHT SEE

Level 4



This student wrote his answers on the question sheet. The coordinates of the points on the grid were given without error. (In fact, the student, while working on the first draft of this activity shown above, discovered and corrected an error in the coordinates of A.) The student also applied the Pythagorean relationship to obtain the correct lengths of line segments in all cases. All the perimeters were calculated correctly. In addition to the flawless application of the Pythagorean relationship, this student was able to find locations for B and C that yield ΔABC of smaller perimeter than those of Andrea and Neeraj, although slightly greater than the optimal triangle. The student also indicates that calculating distances on a grid yields more accurate results than measurement.

Activity 3 – Teacher Edition

Shannon's Secret Revealed

Expectations Addressed

- G 8-5 investigate the Pythagorean relationship using area models and diagrams.
- G 8-8 construct line segments and angles using a variety of methods (e.g. paper folding, ruler and compasses).
- G 8-10 apply the Pythagorean relationship to numerical problems involving area and right triangles.

Context

In Activities 1 and 2, students attempted by trial-and-error to locate grid points B and C on Front Street and Main Street that would minimize the perimeter of ΔABC. In this activity, students use properties of reflections to find the optimal locations for grid points B and C. This builds on the transformational geometry expectations for Grade 7 delineated in *The Ontario Curriculum, Grades 1–8: Mathematics.* The exploration involving reflections also supports the unit on *Optics* in the Grade 8 expectations of the Science and Technology Curriculum. The technique, that we call *Shannon's secret*, is actually an application of a special case of Fagnano's Theorem (see the proof on page 81). Fagnano's Theorem asserts that to find the triangle of smallest perimeter inscribed in a given triangle you need only construct the three altitudes of the given triangle. The feet of these altitudes are the vertices of the inscribed triangle of smallest perimeter. [Your students do not need to know this, but occasionally a student will ask why this technique works. While the discussion on page 81 is mathematically rigorous, students will be more convinced by completing the *Geometer's Sketchpad* Activity (presented in the template on page 80) and then watching the perimeter display as points B and C are moved along DE and OE.]

Activity 3 – Student Page

Shannon's Secret Revealed

In Activity 2, you may have discovered that there are many different ways of choosing grid points B and C to form ΔABC. Therefore computing the perimeter of ΔABC for all possible choices of B and C is not a practical way to find the choice that yields the triangle of smallest perimeter.

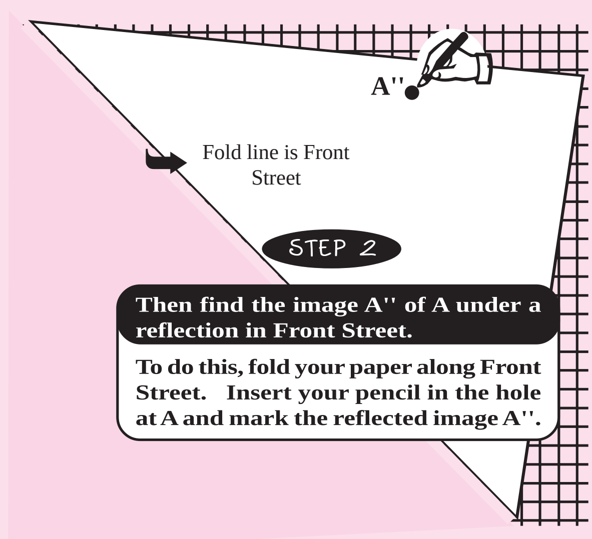
Shannon has a secret *geometric* way of finding points B and C that yield ΔABC of smallest perimeter. To discover her method, get template B from your teacher.

FOLLOW THESE STEPS TO LOCATE POINTS B AND C.

STEP 1

First, find the image A' of A under a reflection in Main Street.

To do this, make a hole in your paper through A. Fold your paper along Main Street. Insert your pencil in the hole and mark the reflected image A'.



STEP 2

Then find the image A'' of A under a reflection in Front Street.

To do this, fold your paper along Front Street. Insert your pencil in the hole at A and mark the reflected image A''.

To locate grid points B and C, students first reflect point A in Main Street, by folding template B (see p. 93) along Main Street or by using a semi-transparent plastic mirror. They label this image A'. Then they reflect A in Front Street to obtain the reflected image that they label as A". They join A' and A". Points B and C are the points where line segment A'A" intersects DE and OE respectively. Having located points B and C on the grid, students record the coordinates of A, B, and C (see Exercise ❶). In ❷, students apply the procedure they learned in Activity 2 for calculating the distance between two grid points. This enables them to determine whether the triangle they obtained in Activity 2 was optimal.

Activity 3 – Teacher Edition

The Lesson Launch

10 minutes

Remind students that the trial-and-error method of finding the optimal triangle is both tedious and time-consuming and offers no guarantee that it will yield the best possible result. For this reason, a direct way of finding the optimal solution is required. Before students embark on this activity, they will need to have an understanding of how to find the reflected image of a point in a line by paper folding and/or by using a semi-transparent plastic mirror. Draw a line on the overhead projector. Then draw a point that is not on the line. Label the point P. Ask students,

- *Suppose I reflect the point P in this line. Where will its image P' appear? Can someone draw it for me?*

Invite a student to make a dot at P'. Ask the student to fold the transparency along the line. Display the folded transparency on the overhead showing P and P' approximately coincident.

- *How did you know where to place the image?*
- *Which point, P or P', do you think is farther from the line? Explain.*
- *At what angle do you think the line segment PP' intersects the line? Give a reason for your answer.*

Allow the students to struggle with the last question before formalizing the relationships between P and P'.

Individual Activity

25 minutes

Distribute template B (see p. 93) and pages 78 and 79. If plastic mirrors are available, allow students to create their reflected images using either a mirror or the paper folding. Explain to students that they have about 25 minutes to do Activity 3 and record their answers to Exercises ❶ through ❹. If you have any dynamical geometry software such as the *Geometer's Sketchpad* or *Cabri Geometry*, distribute the template *Using Sketchpad to Apply Shannon's Secret* (see p. 80) to students who complete the exercises ahead of the rest.

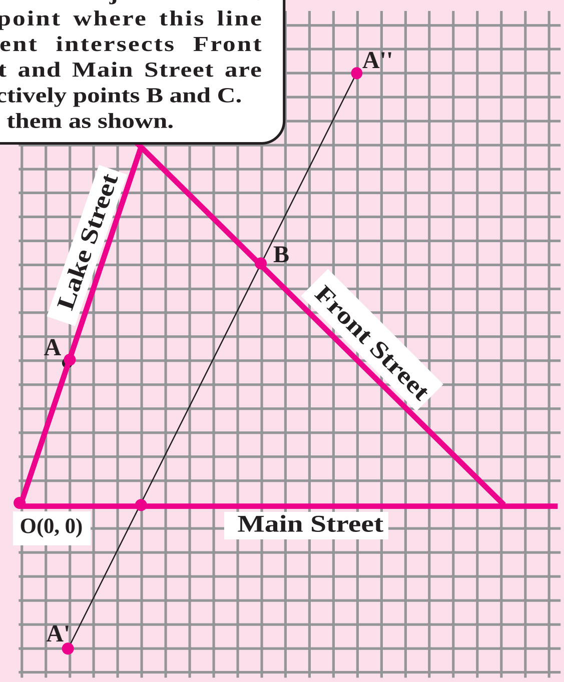
As students complete the Activity, check their answers to Exercise ❸ a) to determine how close they came to discovering the optimal triangle in Activity 2.

Activity 3 – Student Page

STEP 3 — *Shannon's Secret Revealed*

Locate points B and C.

To do this, draw a line segment with a ruler to join A' to A''. The point where this line segment intersects Front Street and Main Street are respectively points B and C. Label them as shown.



❶ After following steps 1 through 3, record the coordinates of A, B, and C relative to Main Street as the X-axis and O as the origin.

❷ Use the Pythagorean relationship to calculate these lengths.
a) |AB| _______ b) |BC| _______ c) |CA| _______
d) perimeter ΔABC ____

❸ a) Is the perimeter of ΔABC less than the perimeter of the ΔABC you obtained in Exercise ❻ of Activity 2?
b) Do you think Shannon's method gives the ΔABC with the smallest perimeter? Explain why or why not.

❹ Shannon folded her grid along the lines of reflection to obtain image points A' and A". Explain how you could locate points A' and A" by counting grid points along lines perpendicular to the reflection lines.

Closure

Discuss with the students their answers to Exercise ❷. Ask students, *What are the values of* $|AB|^2$*?*, $|BC|^2$*?*, $|CA|^2$*? How are these values related?* When students discover that $|AB|^2 + |CA|^2 = |BC|^2$, ask them what this tells us about ΔABC. Students will need to be reminded that the Pythagorean relationship not only implies that the square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides, but conversely, if such a relationship exists among the sides of a triangle, then it is a right triangle. Ask students which side of ΔABC is the hypotenuse and which angle is a right angle. When discussing the answer to Exercise ❸, ask any students who have used dynamical geometry software to discuss their findings.

SHANNON'S SECRET REVEALED

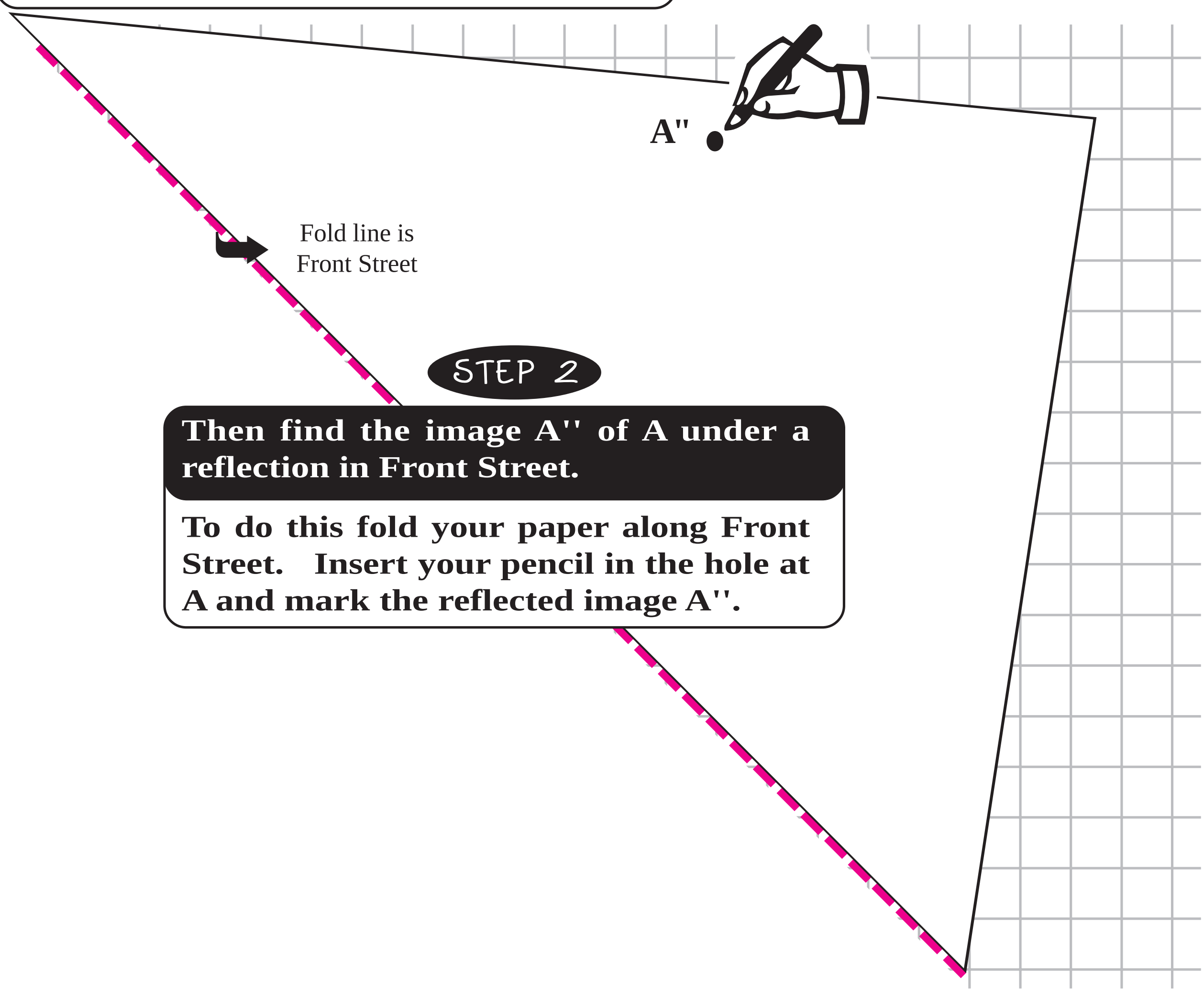
In Activity 2, you may have discovered that there are many different ways of choosing grid points B and C to form ΔABC. Therefore, computing the perimeter of ΔABC for all possible choices of B and C is not a practical way to find the choice that yields the triangle of smallest perimeter.

Shannon has a secret *geometric* way of finding points B and C that yield ΔABC of smallest perimeter. To discover her method, get template B from your teacher.

STEP 1

First, find the image A' of A under a reflection in Main Street.

To do this make a hole in your paper through A. Fold your paper along Main Street. Insert your pencil in the hole and mark the reflected image A'.



STEP 2

Then find the image A'' of A under a reflection in Front Street.

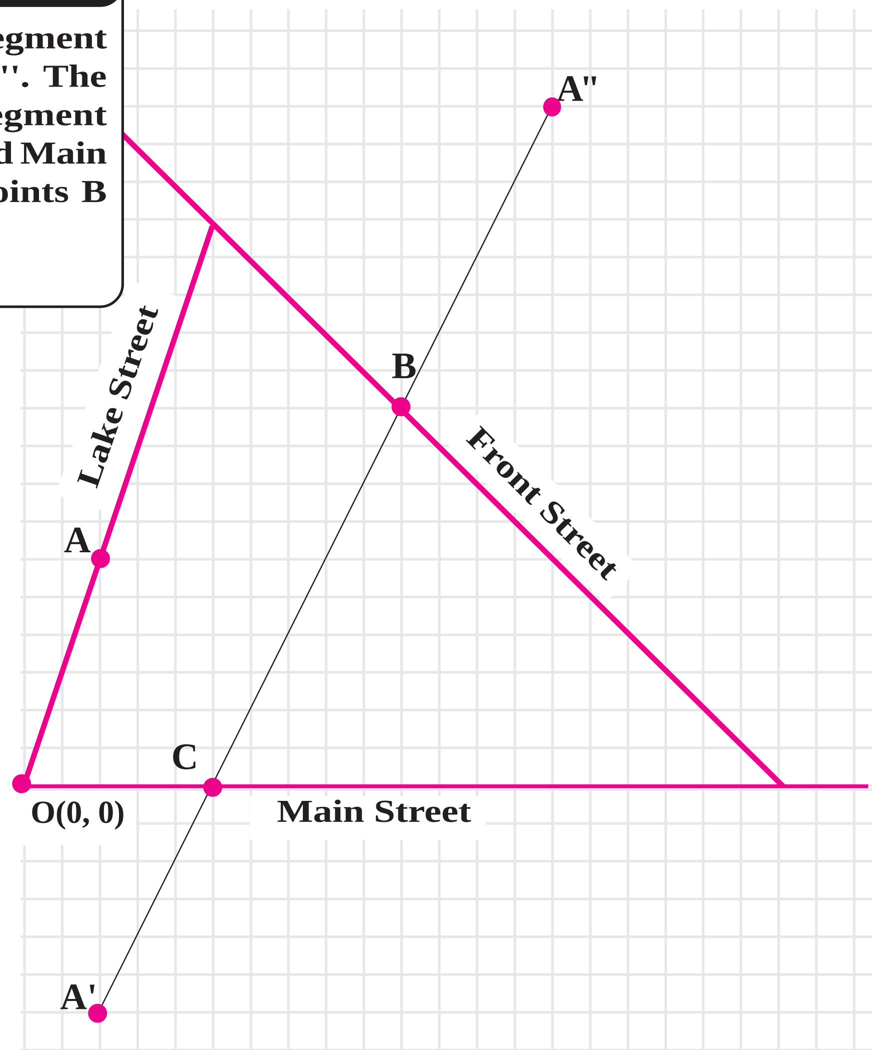
To do this fold your paper along Front Street. Insert your pencil in the hole at A and mark the reflected image A''.

SHANNON'S SECRET REVEALED

STEP 3

Locate points B and C.

To do this, draw a line segment with a ruler to join A' to A''. The points where this line segment intersects Front Street and Main Street are respectively points B and C.
Label them as shown.



❶ After following steps 1 through 3, record the coordinates of A, B, and C relative to Main Street as the X-axis and O as the origin.

❷ Use the Pythagorean relationship to calculate these lengths.
a) |AB| _______ b) |BC| _______ c) |CA| _______ d) perimeter ΔABC ____

❸ a) Is the perimeter of ΔABC less than the perimeter of the ΔABC you obtained in Exercise ❻ of Activity 2?
b) Do you think Shannon's method gives the ΔABC with the smallest perimeter? Explain why or why not.

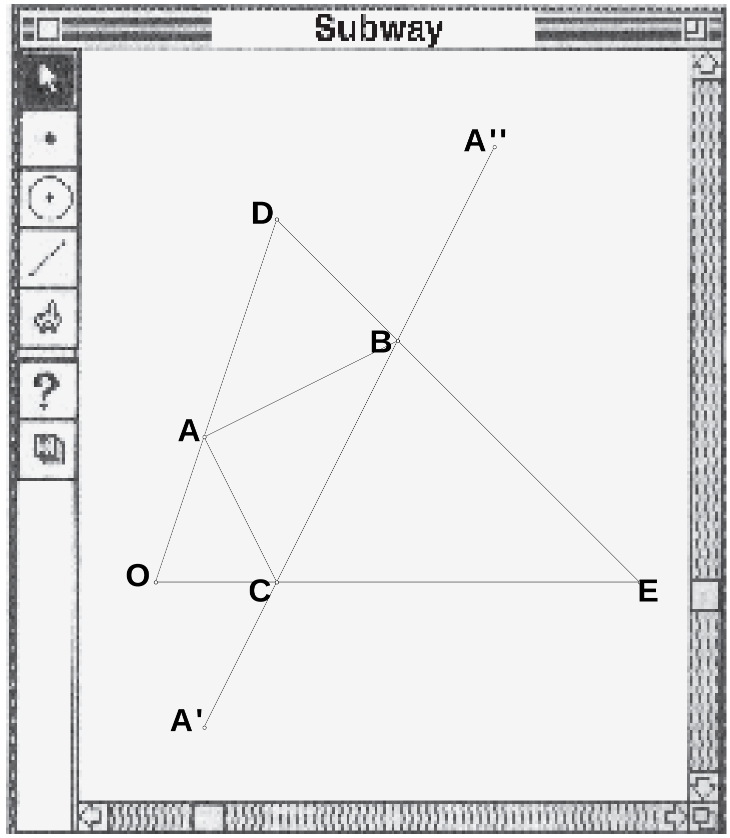
❹ Shannon folded her grid along the lines of reflection to obtain image points A' and A". Explain how you could locate points A' and A" by counting grid points along lines perpendicular to the reflection lines.

TEMPLATE

Using Sketchpad to Apply Shannon's Secret

In Activity 1, you used the *Transform* menu of the *Geometer's Sketchpad* to perform translations of points. In this activity, you will use the same menu to reflect a point in two different lines. This will enable you to apply Shannon's method directly and verify your previous estimate of the minimum perimeter of ΔABC.

To apply Shannon's method, you will find the images A' and A" of the point A under reflections in OE and DE respectively. Then you will construct the line segment A'A". The points where line segment A'A" intersects DE and OE are points B and C respectively. ΔABC is the triangle of minimum perimeter.



1 Construct ΔODE

Proceed as in the first *Sketchpad* activity to construct ΔODE.

2 Construct point A

Construct point A on OD by creating the translation image of O under a translation 2 cm right and 6 cm up, as in the first *Sketchpad* activity.

3 Construct image A' of A under a reflection in OE

Select point A and line segment OE.
From the *Transform* menu select *Mark Mirror "m."*
From the *Transform* menu select *Reflect.*
Label the reflected image A'.

Tool/Menu

Select A and OE.
Transform –> Mark Mirror "m"
Transform –> Reflect
Select Text Tool.
Double-click Label and enter A'.

4 Construct image A'' of A under a reflection in DE

Select point A and line segment DE.
Proceed as in step 3 to obtain image A under a reflection in line segment DE.
Label the image A".

Select A and DE.
Transform –> Mark Mirror "n"
Transform –> Reflect
Select Text Tool.
Double-click Label and enter A".

5 Display the perimeter of ΔABC

Construct line segments AB and AC.
Select points A, B, and C.
Construct the interior of ΔABC.
Display the perimeter of ΔABC.

Select points A, B, and C.
Construct –> Polygon Interior
Measure –> Perimeter

Grade 8
Answer Key for Activity 3

❶ The coordinates are A(2, 6), B(10, 10), C(5, 0).

❷ a) $|AB| = \sqrt{64+16} \approx 8.9$

b) $|BC| = \sqrt{25+100} \approx 11.2$

c) $|CA| = \sqrt{9+36} \approx 6.7$

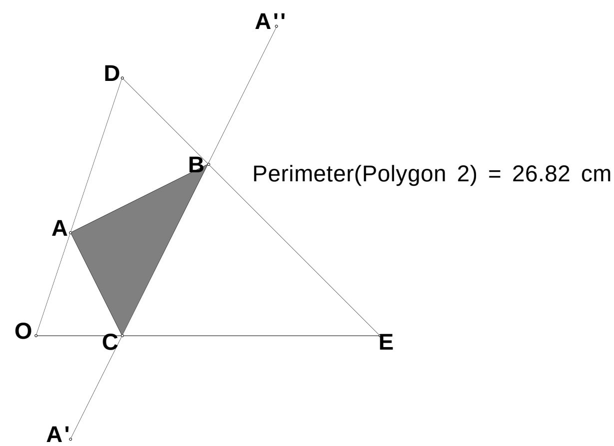
d) Perimeter ΔABC = 26.8

❸ a) Since Shannon's method yields the triangle of smallest perimeter, the student's triangle obtained in Activity 2 will have perimeter equal to or greater than ΔABC obtained using Shannon's method. The latter case is more likely.

b) Students will have no way of knowing whether Shannon's method yields the triangle of smallest perimeter. However, by comparing the perimeter of that triangle with the perimeters of the triangles they obtained in Activity 2, most students will be satisfied that Shannon's method yields the optimal triangle.
An elegant way to see why Shannon's method works is presented below for your interest, but it is beyond most students at this level.

❹ The line perpendicular to Front Street is a diagonal line passing through the grid points. It passes through 12 grid points between A' and Front Street. Therefore we obtain its reflected image, A'', by counting 12 grid points past Front Street along the perpendicular to Front Street.

Using Sketchpad to Apply Shannon's Secret

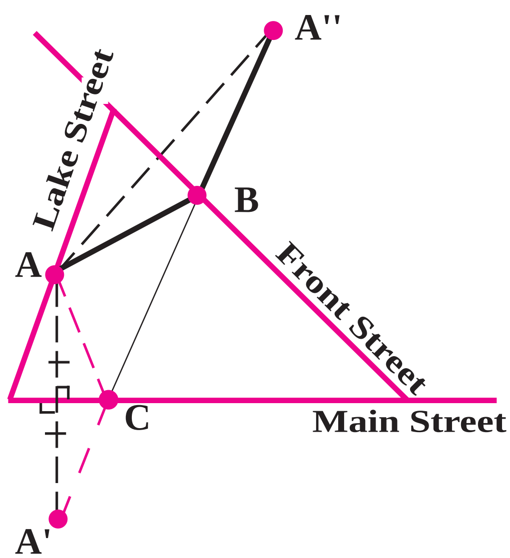


The Secret Behind Shannon's Secret

The secret behind Shannon's secret is the discovery by J. F. Toschi di Fagnano in 1775, known as Fagnano's Theorem. A special case of this theorem asserts that if A is a given point on Lake Street (see diagram), then the triangle ABC of smallest perimeter that can be inscribed in the triangle formed by the three streets has vertices B and C on segment A'A'', where A' and A'' are the reflected images of A in Main Street and Front Street respectively. To see why this is true, we observe that since A' is the reflected image of A, AC = A'C. Similarly, BA = BA''. Therefore,

$$\underbrace{AC + CB + BA}_{\text{Perimeter of } \Delta ABC} = \underbrace{A'C + CB + BA''}_{\text{Distance from A' to A'' along the path } A' \to C \to B \to A''}$$

The distance from A' to A'' along the path A' → C → B → A'' is shortest when B and C lie on segment A'A'', so the perimeter of ΔABC is a minimum for this choice of B and C.



Activity 4 – Teacher Edition
Shannon Plays the Angles

Expectations Addressed

G 8-3 identify the angle properties of intersecting, parallel, and perpendicular lines by direct measurement: interior, corresponding, opposite, alternate, supplementary, complementary.

G 8-4 explore the relationship to each other of the internal angles in a triangle (they add up to 180˚) using a variety of methods (e.g., aligning corners of a paper triangle, using a protractor).

G 8-5 investigate the Pythagorean relationship using area models and diagrams.

G 8-6 solve angle measurement problems involving properties of intersecting line segments, parallel lines, and transversals.

G 8-8 construct line segments and angles using a variety of methods (e.g. paper folding, ruler and compasses).

G 8-10 apply the Pythagorean relationship to numerical problems involving area and right triangles.

Context

The expectations in the Grade 8 Geometry and Spatial Sense strand, as defined on page 51 of *The Ontario Curriculum, Grades 1–8: Mathematics*, focus on the development of relationships among angles. Activity 4 addresses particularly those expectations that pertain to the relationships between angles in a triangle. In Exercise ❸ a and b, students use the converse of the Pythagorean Theorem to determine that ΔOBE is a right triangle and the ∠OBE is a right angle. (This reinforces the work of Activity 3 in which they proved the analogous result for ΔABC.) Then in Exercise ❸ c, students apply the Isosceles Triangle Theorem to deduce that ∠BOE = ∠BEO. In Exercise ❸ d, students are led to deduce (from the sum-of-the-angles-in-a-triangle relationship) that ∠BOE = ∠BEO = 45˚. Students apply this relationship again in Exercise ❸ e to deduce that ∠CBE has measure 135˚ – *x*. Exercise ❹ involves the students in tracing and cutting out ΔBDA and placing it over ΔBCE so that the two triangles coincide along two sides and have parallel bases. Using the fact that ∠BDA and ∠BCE are corresponding angles, or by measurement with a protractor, students conclude that ∠BDA = ∠BCE, and so ΔBDA and ΔBCE are equiangular. From this, students conclude that the measure of ∠BAD is 45˚. In Exercise ❺, students deduce that ∠CAB is a right angle. This result is used in Exercise ❻ to show that ΔCOA and ΔCBE are equiangular. Measuring ∠BCE ≈ 63˚ enables students to obtain a value for *x* that they can substitute into the expressions for the measures of other angles. Students can verify that all three triangles, ΔCOA, ΔCBE, and ΔDBA are equiangular!

Activity 4 – Student Page

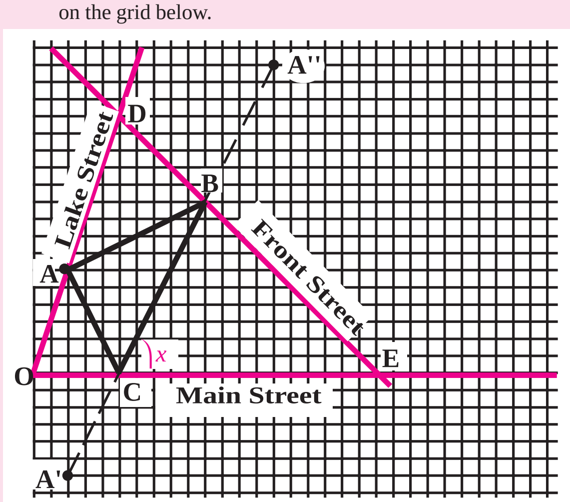
Shannon Plays the Angles

Shannon's geometric method yields the ΔABC shown below.

❶ Using point O as the origin and Main Street as the X-axis, write the coordinates of these locations (where D and E are the intersections where the major streets meet).
a) A(___ , ___) b) B(___ , ___) c) C(___ , ___)
d) D(___ , ___) e) E(___ , ___) f) A''(___ , ___)

❷ Compare your answers in Exercise ❶ above with your answers to Exercise ❶ in Activity 3. If your answers are different, check your work in Activity 3.

❸ To construct the subway tunnel, Shannon must first determine the measures of the angles in ΔBCE. Help her by answering these questions without measuring any angles.
Let *x* denote the measure of ∠BCE in degrees (as shown below).
a) Calculate the lengths of OB, BE and OE. Is ΔOBE a right triangle? Explain.
b) What is the measure of ∠OBE?
c) Is OB = BE? What does this tell you about ∠BOE and ∠BEO?
d) What is the measure of ∠BEC in degrees? Mark this on the grid below.
e) Write an expression in *x* for the measure of ∠CBE. Write this on the grid below.



Activity 4 – Teacher Edition

The Lesson Launch 15 minutes

Display on the overhead projector a transparency of the grid on page 84. Ask students questions such as the following, and write on the transparency the lengths of the line segments as they are discovered.

- *What are the coordinates of points A, B, and C?*
- *What are the lengths of AB, AC, and BC?*
- *How can you determine from these lengths whether ΔABC is a right triangle?*
- *What can you say about the angles opposite the equal sides of an isosceles triangle?*
- *Sketch a right triangle with two equal sides. Mark the right angle. What are the measures of the other two angles of your isosceles triangle? Explain.*

Through discussion, review with students such relationships as:

- the angles opposite equal sides of an isosceles triangle are equal.
- corresponding angles formed by a transversal intersecting two parallel lines have equal measure.
- the sum of the measures of the interior angles of a triangle is 180˚.

Ensure that students are able to use the latter relationship to express the measures of the unknown angles in a right triangle as x and $90˚ – x$.
Distribute student pages 84 and 85. Have students complete individually Exercises ❶ and ❷.

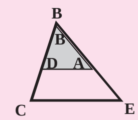
Paired Activity 25 minutes

Group students in pairs. Provide each pair with a protractor. Have them complete Exercises ❸ through ❻ in their notebooks and record the angle measures on page 84. (If you have any dynamical geometry software such as the *Geometer's Sketchpad* or *Cabri Geometry*, distribute the template *Using Sketchpad to Measure Angles and Distance* (see p. 86) to students who complete the exercises ahead of the rest.) When all pairs have completed Exercises ❸ through ❻, display your transparency of the grid on page 84 and invite student answers to each of the exercises. As students announce the measures of various angles, have them come to the overhead projector and record the measures of the angles they have obtained. Ask them to explain, using the transparency, how they obtained their answers.

Activity 4 – Student Page

Shannon Plays the Angles

❹ Next we must find the measures of the angles in ΔBDA. To do this, follow these steps.
a) Trace ΔBDA, label its vertices on the tracing and then cut it out.
b) Place your tracing of ΔBDA on top of ΔBCE so that vertex B on the tracing falls on vertex B of your grid.
c) Are ΔBDA and ΔBCE equiangular?
i.e., Is ∠BDA = ∠BCE? and is ∠BAD = ∠BEC? Explain.
d) Write on your grid the measure of ∠BAD.
Write the measure of ∠BDA in terms of x.



❺ Do you think ∠CAB is a right angle? Calculate the lengths of AB, AC and BC to verify your answer. Record the measure of ∠CAB on your grid.

❻ a) Use the measures of the angles marked on your grid to calculate the measures of the three angles in ΔCOA. Are ΔCOA and ΔCBE equiangular? Explain how you know.
b) Use your protractor to measure ∠BCE. Use this measure to obtain the measures of ∠ABC and ∠ACB in degrees.
c) List the measures of all three angles in ΔABC and check that they total 180˚.

❼ Why didn't Shannon just measure the angles in ΔABC instead of performing all these calculations?

Research Report

Write a report of a half page or more to the City Planner, Mr. Ofoscu, describing your recommendations for the locations of stations B and C.

Support your recommendation by including the following elements in your report:

- the precise coordinates of stations A, B, and C
- the reason why you believe that this choice of locations will result in the lowest possible cost.
- the measures of the three angles within ΔABC.
- possible reasons why the most economical choice of locations B and C might not be the most desirable choice for the city.
- a copy of the grid showing the locations of B and C and the appropriate angle measures.

Closure

Discuss Exercise ❼ with the students. Guide them to the realization that each measurement involves some error because of the limited precision of the measuring instrument. When solving problems in real-world situations, we can minimize such errors by using deductions wherever possible to reduce the number of measurements needed.

Have students work individually on the research report outlined on page 85. If you have access to dynamical geometry software such as the *Geometer's Sketchpad*, you can provide students with the option to complete instead the templates on pages 64-65, 80, and 86, and then write a report on their findings.

Shannon Plays the Angles

Shannon's geometric method yields the ΔABC shown below.

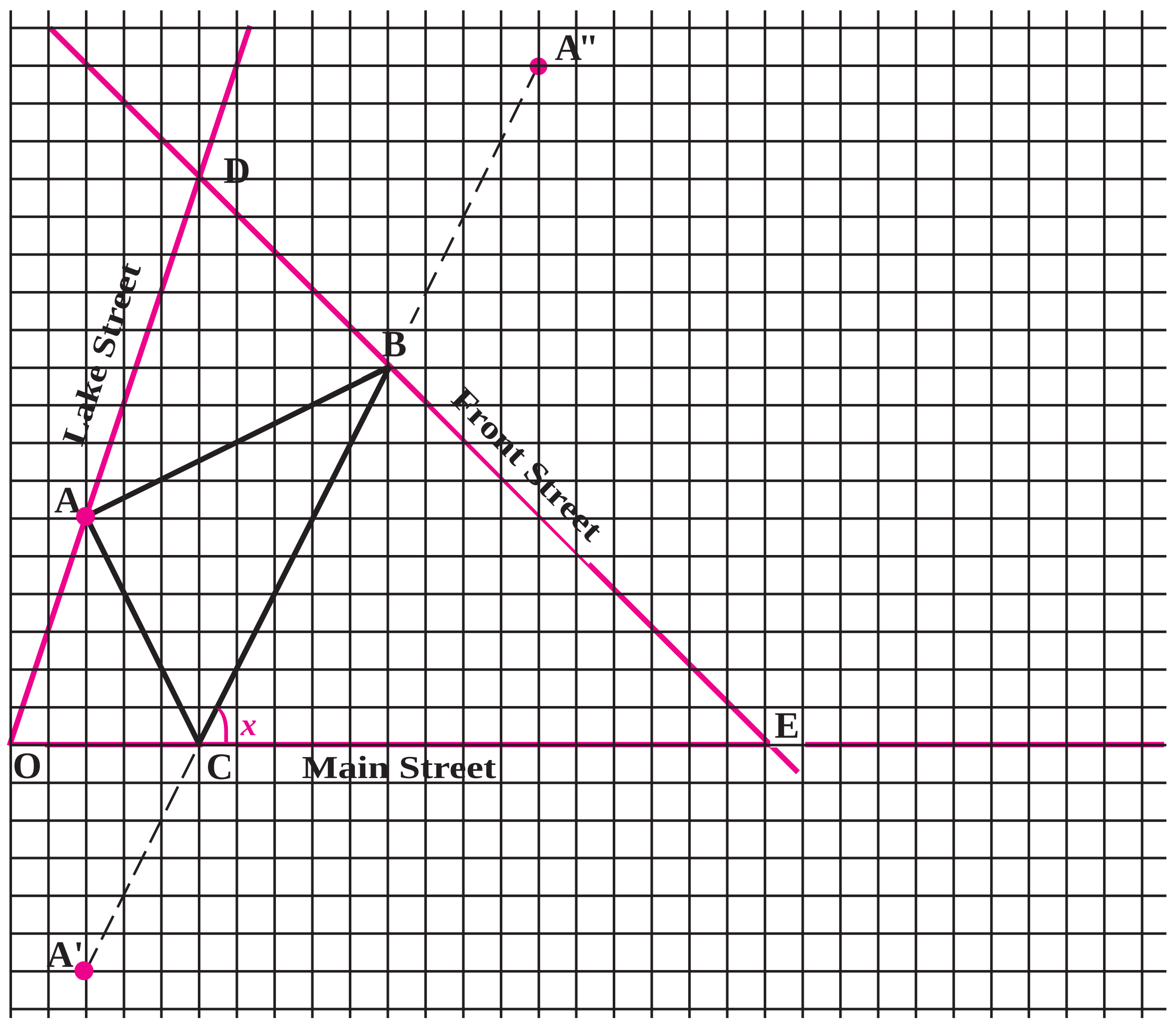
❶ Using point O as the origin and Main Street as the X-axis, write the coordinates of these locations (where D and E are the intersections where the major streets meet).

a) A(___ , ___)   b) B(___ , ___)   c) C(___ , ___)
d) D(___ , ___)   e) E(___ , ___)   f) A"(___ , ___)

❷ Compare your answers in Exercise ❶ above with your answers to Exercise ❶ in Activity 3. If your answers are different, check your work in Activity 3.

❸ To construct the subway tunnel, Shannon must first determine the measures of the angles in ΔBCE. Help her by answering these questions without measuring any angles. Let x denote the measure of ∠BCE in degrees (as shown below).

a) Calculate the lengths of OB, BE and OE. Is ΔOBE a right triangle? Explain.
b) What is the measure of ∠OBE?
c) Is OB = BE? What does this tell you about ∠BOE and ∠BEO?
d) What is the measure of ∠BEC in degrees? Mark this on the grid below.
e) Write an expression in x for the measure of ∠CBE. Write this on the grid below.



Shannon Plays the Angles

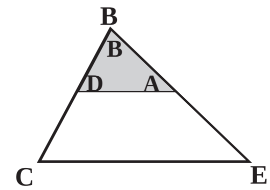
❹ Next we must find the measures of the angles in ΔBDA. To do this, follow these steps.

a) Trace ΔBDA, label its vertices on the tracing and then cut it out.

b) Place your tracing of ΔBDA on top of ΔBCE so that vertex B on the tracing falls on vertex B of your grid as shown.

c) Are ΔBDA and ΔBCE equiangular?
i.e., Is ∠BDA = ∠BCE? and is ∠BAD = ∠BEC? Explain how you know.

d) Write on your grid the measure of ∠BAD.
Write the measure of ∠BDA in terms of *x*.



❺ Do you think ∠CAB is a right angle? Calculate the lengths of AB, AC and BC to verify your answer. Record the measure of ∠CAB on your grid.

❻ a) Use the measures of the angles marked on your grid to calculate the measures of the three angles in ΔCOA. Are ΔCOA and ΔCBE equiangular? Explain how you know.

b) Use your protractor to measure ∠BCE. Use this measure to obtain the measures of ∠ABC and ∠ACB in degrees.

c) List the measures of all three angles in ΔABC and check that they total 180˚.

❼ Why didn't Shannon just measure the angles in ΔABC instead of performing all these calculations?

Research Report

Write a report of a half page or more to the City Planner, Mr. Ofoscu, describing your recommendations for the locations of stations B and C.

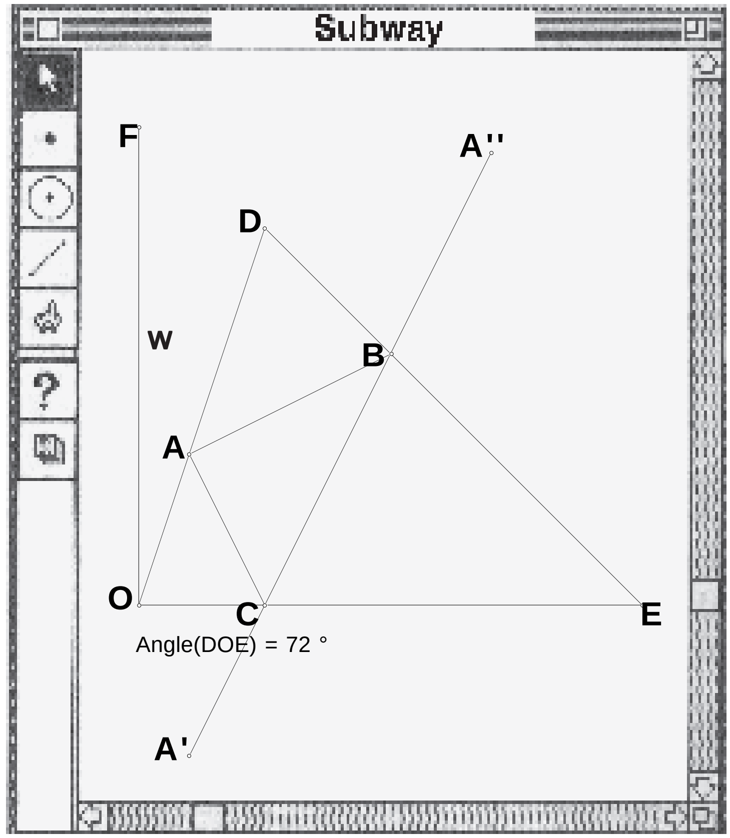
Support your recommendation by including the following elements in your report:

- the precise coordinates of stations A, B, and C
- the reason why you believe that this choice of locations will result in the lowest possible cost.
- the measures of the three angles within ΔABC.
- possible reasons why the most economical choice of locations B and C might not be the most desirable choice for the city.
- a copy of the grid showing the locations of B and C and the appropriate angle measures.

TEMPLATE

Using Sketchpad to Measure Angles and Distance

In the previous activities, you used the *Measure* menu of the *Geometer's Sketchpad* to measure the perimeter of ΔABC. In this activity, you will use the Measure menu to measure angles and distances.



❶ Construct ΔODE, ΔABC, A' and A" as in the previous *Sketchpad* activities.

❷ Construct line segment OF, where F has coordinates (0, 14), to serve as a positive Y-axis. (See diagram on the right.)

❸ a) Select *in order* the points D, O, and E. Then select *Angle* from the *Measure* menu.
You will see displayed the measure of ∠DOE.
b) Find the measures of ∠DEO and ∠ODE.
c) What is the sum of the angles in ΔDOE?

❹ Using the same procedure as in ❸, find the measures of the three angles in ΔABC.
a) What is the sum of the angles in ΔABC?
b) Is ΔABC a right triangle?
c) Select line segment AB, and select *Length* from the *Measure* menu to determine its length.
d) Measure the lengths of line segments BC and AC and verify that ΔABC satisfies the Pythagorean relationship.

❺ a) Select line segment OF and point A". Then select *Distance* from the *Measure* menu. The distance of A" from the y-axis OF will be displayed.
b) Measure the distance of point A" from OE.
c) Write the coordinates of point A".

❻ a) Use the method in Exercise ❺ to determine the coordinates of A'.
b) Measure the distance from A' to A".
c) Compare the distance between A' and A" with the perimeter of ΔABC. Describe what you discover.

Grade 8
Answer Key for Activity 4

❶ The coordinates are:

a) A(2, 6) b) B(10, 10) c) C(5, 0)
d) D(5, 15) e) E(20, 0) c) A"(14, 18)

❸ a) $|OB| = \sqrt{10^2 + 10^2} = \sqrt{200}$

$|BE| = \sqrt{10^2 + 10^2} = \sqrt{200}$

$|OE| = 20$

Since $|OE|^2 = |OB|^2 + |BE|^2$, then ΔOBE is a right triangle.

b) It follows from ❸ a) that ∠OBE = 90˚.

c) From ❸ a) we observe that |OB| = |BE|, so ∠BOE = ∠BEO. (Isosceles Triangle Theorem)

d) Since ΔOBE is a right triangle,
∠BOE + ∠BEO= 90˚
But, ∠BOE = ∠BEO
Therefore ∠BOE = ∠BEO = 45˚.
Since ∠BEO = ∠BEC, ∠BEC = 45˚.

e) Since the sum of the angles in a triangle is 180˚,
∠CBE + ∠BEC + x = 180˚.
∠CBE + 45˚ + x = 180˚.
Therefore ∠CBE = 135˚ – x.

❹ c) At this level it is acceptable for students to observe that BD appears to be fall along BC and BA fall along BE. From this, they can conclude that ∠DBA = ∠CBE. By sliding ΔBDA downward, students can verify that ∠BDA = ∠BCE and ∠BAD = ∠BEC. That is, ΔBDA and ΔBCE are equiangular.

d) ∠BAD = ∠BEC = 45˚.
Since ∠BDA = ∠BCE, then ∠BDA = x.

❺ $|AB|^2 = 8^2 + 4^2 = 80$
$|AC|^2 = 6^2 + 3^2 = 45$
$|BC|^2 = 10^2 + 5^2 = 125$
Since $|AB|^2 + |AC|^2 = |BC|^2$, then ΔABC is a right triangle and ∠CAB = 90˚.

Teacher Note

In this unit we have not distinguished between an angle and its measure. Strictly formal notation requires that we write angle measures as follows: measure (∠ABC) = 36˚. In the interest of simplicity, we have used the abbreviated notation ∠ABC = 36˚.

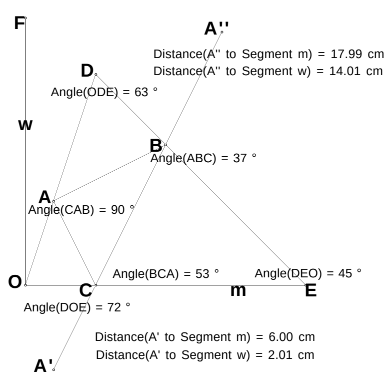
❻ a) ∠OAC = 180˚ – ∠DAB – ∠CAB.
= 180˚ – 45˚ – 90˚.
= 45˚.
∠AOC = 180˚ – ∠DEO – ∠ODE.
= 180˚ – 45˚ – x˚.
= 135˚ – x.
∠ACO = 180˚ – ∠OAC – ∠AOC.
= 180˚ – 45˚ – (135˚– x)
= x˚.
Therefore ΔCOA and ΔCBE are equiangular.

b) Using a protractor, we find ∠BCE ≈ 63˚. (A more precise measure is 63.43˚.) That is, x ≈ 63.
Therefore ∠ACB = 180˚ – ∠ACO – ∠BCE
= 180˚ – x˚ – x˚
≈ 180˚ – 63 – 63
≈ 54˚ (actually closer to 53˚)
And ∠ABC = ∠BAC – ∠ACB
≈ 90˚ – 54˚
≈ 36˚ (actually closer to 37˚)

c) The measures of the three angles in ΔABC are:
∠ABC ≈ 36˚; ∠ACB ≈ 54˚; ∠BAC = 90˚.

❼ If the measure of ∠CAB has a small error, it will cause a large error in the location of B, because AB is so long. Every measurement has some amount of error. When several measurements are combined, the cumulative error can be large. To reduce the number of measurements to one, Shannon has calculated the measures of all the angles in terms of ∠BCE. After she has measured that angle with a high precision, she calculates the other angles. As long as the measure of ∠BCE is accurate and precise, the measures of all the angles will be accurate and the points B and C will be located correctly.

Using Sketchpad to Measure Angles & Distance



The scoring guide presented below has been developed using student responses on a field test conducted in 1998. However, it is recommended that you regard it merely as a starting point in the development of your own scoring guide that will evolve as you use this with students. *The Ontario Curriculum, Grades 1–8: Mathematics* asserts:

> *Level 3, the "provincial standard," identifies a high level of achievement of the provincial expectations. Parents of students achieving at Level 3 in a particular grade can be confident that their children will be prepared for work at the next grade.*

For this reason, the scoring guides in this module shade the criteria in the Level 3 column, and on occasion relate other levels to Level 3 achievement.

Scoring Guide for Activity 4

	Level 1	Level 2	Level 3	Level 4
COMMUNICATION **Communication or knowledge of coordinates, angular measure, and optimization concepts** **General Expectation:** Use mathematical language effectively to describe geometric concepts, reasoning, and investigations.	• Report is often incoherent and includes at most one of the following elements without errors. • coordinates of the vertices of ΔABC. • measures of the angles of ΔABC. • reason why the proposed solution is optimal. • reasons why the most economical solution may not be the best. • a copy of the grid showing the measures of the angles of ΔABC.	• Report is mostly coherent and includes two of the following elements with no errors. • coordinates of the vertices of ΔABC. • measures of the angles of ΔABC. • reason why the proposed solution is optimal. • reasons why the most economical solution may not be the best. • a copy of the grid showing the measures of the angles of ΔABC.	• Report is coherent and includes at least three of the following elements with little or no errors. • coordinates of the vertices of ΔABC. • measures of the angles of ΔABC. • reason why the proposed solution is optimal. • reasons why the most economical solution may not be the best. • a copy of the grid showing the measures of the angles of ΔABC.	• Report is coherent and includes at least four of the following elements with little or no errors. • coordinates of the vertices of ΔABC. • measures of the angles of ΔABC. • reason why the proposed solution is optimal. • reasons why the most economical solution may not be the best. • a copy of the grid showing the measures of the angles of ΔABC.

WHAT YOU MIGHT SEE

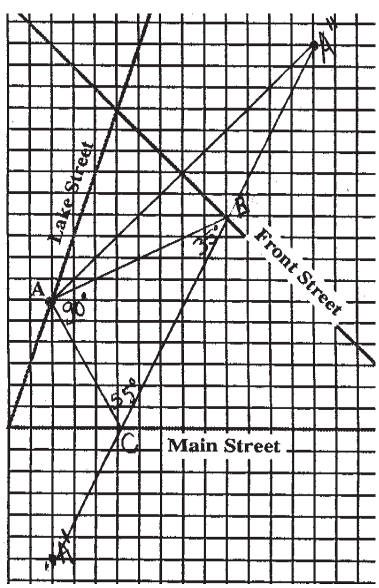
Level 2

Report

Mr. Orosco
I think the locations of stations B and C
are good choices. The precise coordinates of A, B,
and C are as follows: A: (2,6) B: (10,10) C: (5,0)
I think the streets are like this because
they cost cheap to make because they are 2 km apart.
The measures of the angles within ΔABC are as
follows ∠ABC: 90°, ∠ACB: 55°, ∠CAB: 35°. I think that
locations of B and C are a desirable choice for
the city. The grid is included on another page.
You might want to move B and C a little closer
so you can make more Money. Also try
to make a small perimeter as you can. Are
you going to make any more streets like
this? If so what will they be called? How long
does it take to do something like this?
Is this how it was really planned to
be done for all of these streets.
Sincerely,

← Correct coordinates for the vertices of ΔABC.

← Correct measures for the angles of ΔABC, but incorrectly labelled.



This report is mostly coherent. The positions of A" and A' are located correctly and consequently B and C were also located correctly. The student has given the correct coordinates for A, B, and C. Although the correct measures (within the range of permissable error) of the angles of ΔABC were shown on the grid, the student did not name them correctly in his report. The student seems to have understood that the reason to minimize the perimeter of ΔABC is to minimize cost, but he states, "You might want to move B and C closer so you can make more money." Does this mean that the student does not realize that the locations of B and C found by Shannon's process yields the ΔABC of smallest perimeter? The report suggests that the streets are constructed 2 km apart because "they cost cheap."

In essence, the report contains the grid with appropriately marked angle measures and the correct coordinates for the vertices of ΔABC. With a little coaching on the correct way to name angles and on the concept of minimum perimeter, this student could reach Level 3 performance.

WHAT YOU MIGHT SEE

Level 3

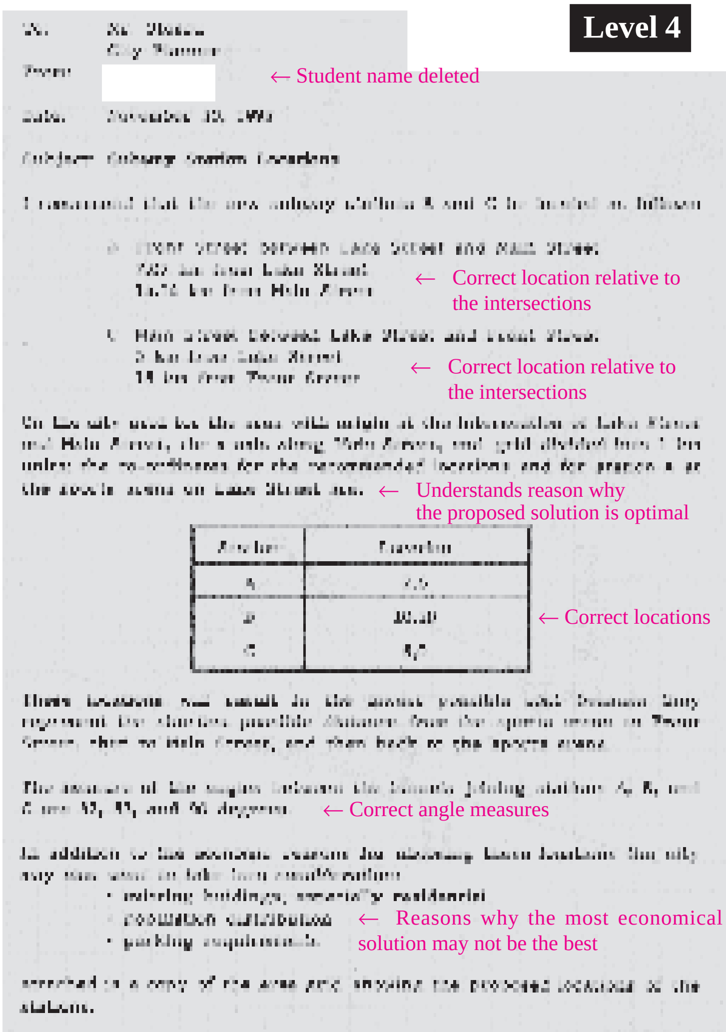
Dear Mr. Sloan,
I have made numerous ways of placing the new upcoming station along Lake, Main, and Front streets. The location should be the one that Shannon came up with because it is the shortest and cheapest route to take. My smallest direction was 27 km while Shannon's was 26.25542 km. For the shortest route, use the following coordinates:
A = (2, 6) B = (10, 10) C (5, 0)
The angles are:
∠A = 90° ∠B = 45° ∠C = 55°

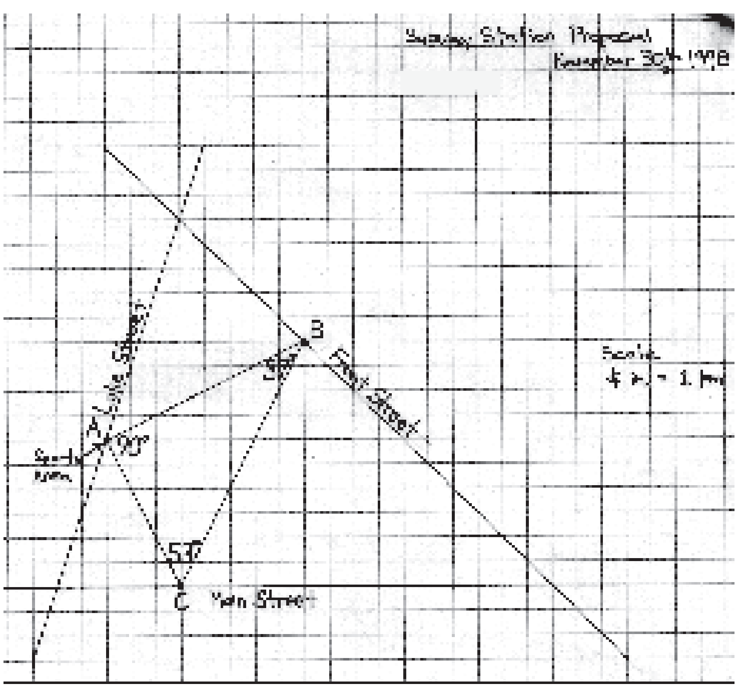
The area of B and C might not be the best spot because the city might want a shorter B and C side/streets for economical or industrial reasons. There also could possibly be a construction at the 2 areas so it could be very inconvenient for the location of points B and C, but hopefully there would be no problems and the station could be built.

This report is coherent although the sentence structure is sometimes awkward. The positions of A" and A' were located correctly and, consequently, B and C were also located correctly. The student has given the correct coordinates for A, B, and C. The correct measures (within the range of permissable error) of ∠A and of ∠ C of ΔABC were given, but the measure given for ∠B was incorrect. The student understands that the route proposed is optimal in that it is “the shortest and cheapest route to take.” The student also suggests that the most economical locations for B and C may not be the most desirable because local business or industry may require sites B and C to be closer together. The report also suggests that the implications of construction at proposed sites B and C must also be considered. Since this report contains the correct locations of the vertices of ΔABC, a reason why the proposed solution is optimal, reasons why the optimal solution may not be best, and a properly labelled grid (not shown here, but mostly correct), it qualifies as Level 3 performance. If the measures of all the angles were correct, and these were shown on the grid, the report would qualify for a Level 4 rating.

WHAT YOU MIGHT SEE

Level 4



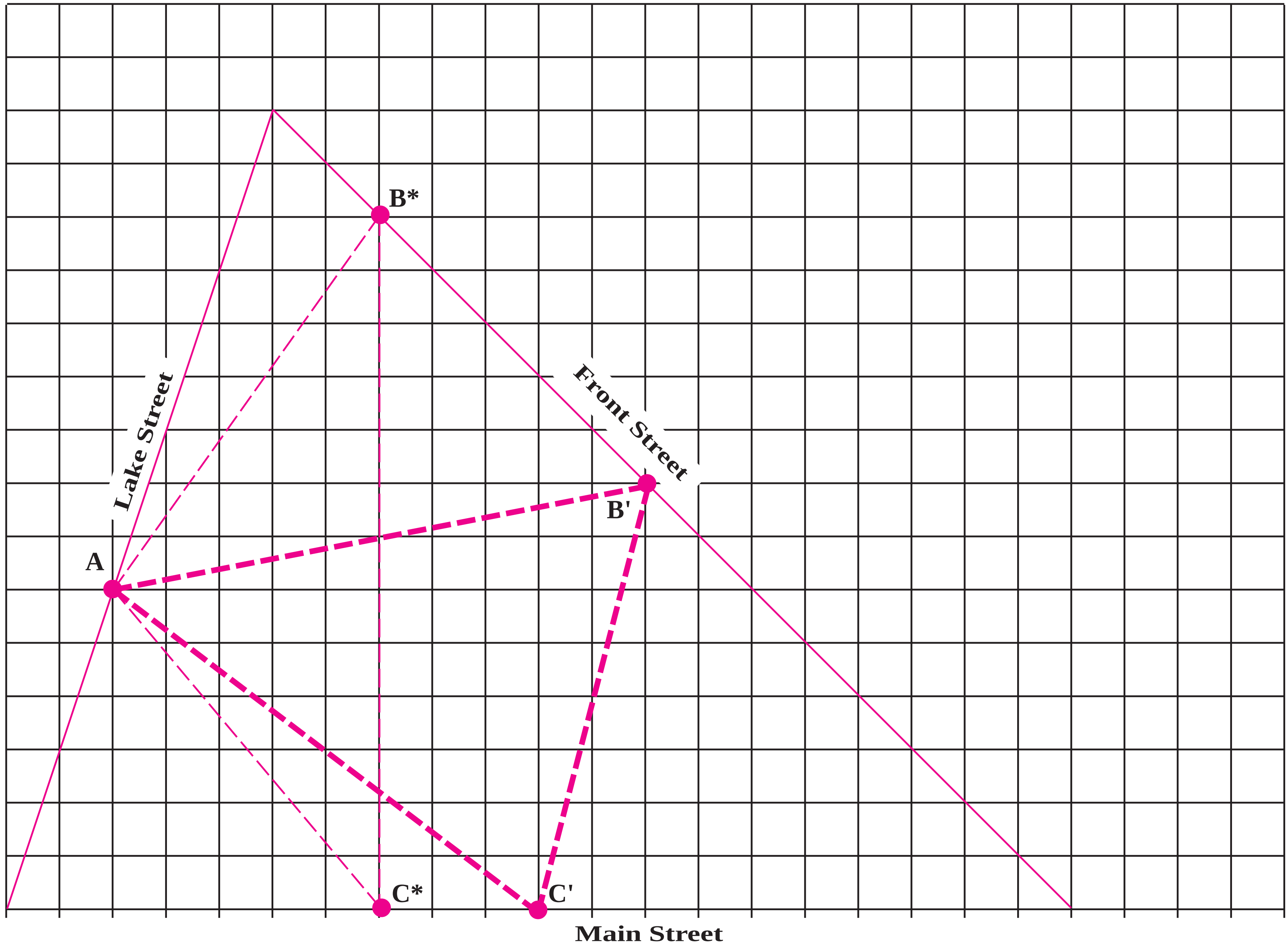


This report is exemplary. The student has used a formal memo format, laid out the information succinctly, and presented a coherent and readable report that would be a credit to a secondary school student. The positions of A" and A' are located correctly and consequently B and C were also located correctly. The student has given the correct coordinates for A, B, and C. The correct measures (within the range of permissable error) of the angles of ΔABC were presented in the report and also shown on the grid included with the report.

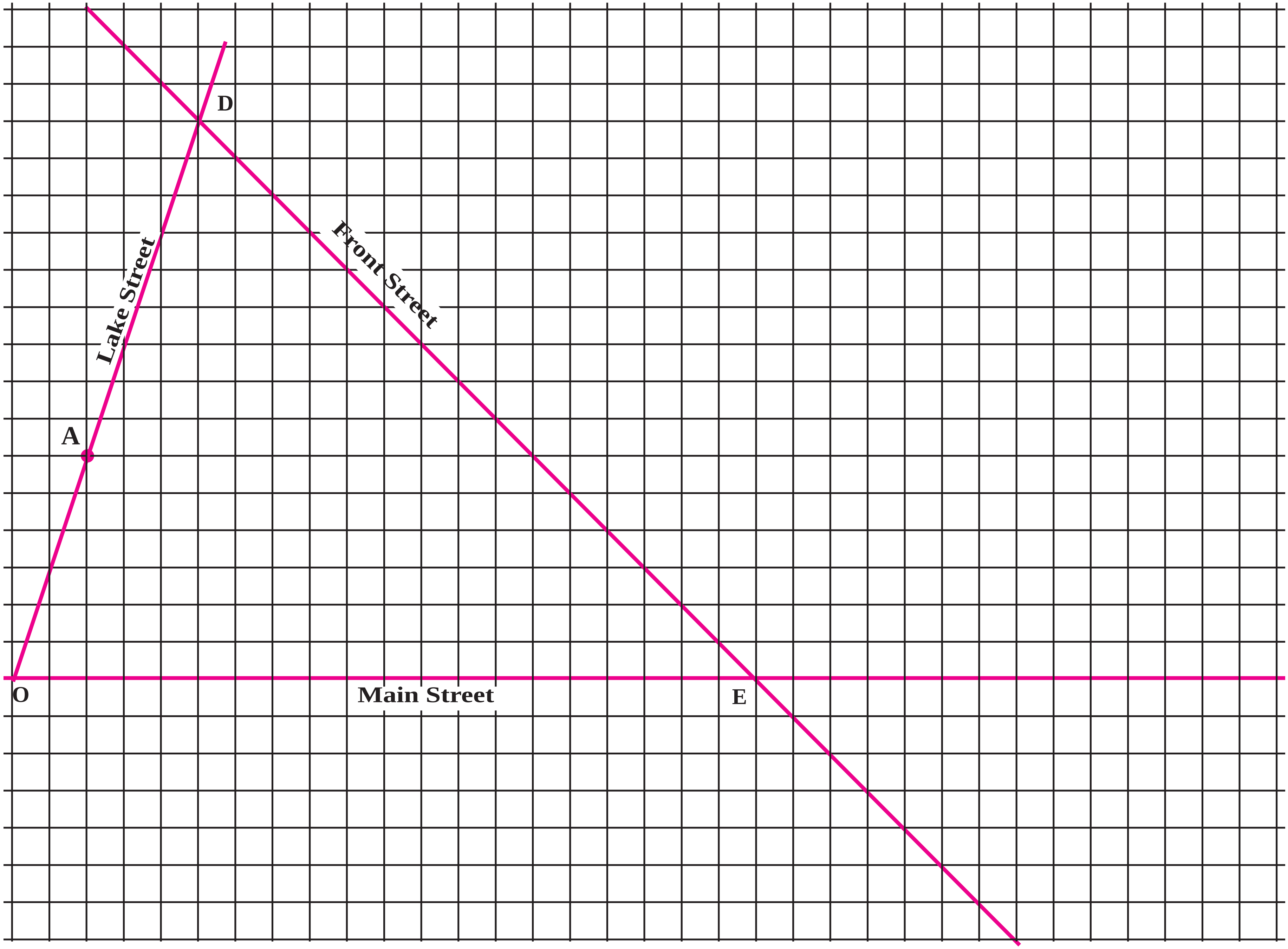
The report states explicitly that the proposed locations may be the most economical because they minimize the distance from the sports arena, to Front Street, to Main Street and back. This reveals an awareness of the meaning of an "optimal solution." The student suggests that the most economical locations for B and C may not be the most desirable, and has included three other issues that must be considered before a final decision is reached.

Such performance as this student has demonstrated, is an inspiration to all of us in the teaching profession. It indicates that at least some of our students are achieving at the highest levels and surpassing our highest expectations. This encourages us to motivate even more of our students to this level of performance. Hats off to teacher and student!

Template A   City Planning: Zoom-In View



Template B City Planning: Zoom-Out View



TEMPLATE

Record of Student Achievement on the Grade 7 Unit

How Many Faces Has Cinesphere?

Student Name ______________________________

From Scoring Guide for Activity 1 p. 29

Summary	Level
Problem Solving	
Concepts	

From Scoring Guide for Activity 2 p. 37

Summary	Level
Application	

From Scoring Guide for Activity 3 p. 45

Summary	Level
Problem Solving	

Combining the Scores from all Scoring Guides

Summary	Level
Problem Solving	
Concepts	
Application	

Student Name ______________________________

From Scoring Guide for Activity 1 p. 29

Summary	Level
Problem Solving	
Concepts	

From Scoring Guide for Activity 2 p. 37

Summary	Level
Application	

From Scoring Guide for Activity 3 p. 45

Summary	Level
Problem Solving	

Combining the Scores from all Scoring Guides

Summary	Level
Problem Solving	
Concepts	
Application	

Student Name ______________________________

From Scoring Guide for Activity 1 p. 29

Summary	Level
Problem Solving	
Concepts	

From Scoring Guide for Activity 2 p. 37

Summary	Level
Application	

From Scoring Guide for Activity 3 p. 45

Summary	Level
Problem Solving	

Combining the Scores from all Scoring Guides

Summary	Level
Problem Solving	
Concepts	
Application	

Student Name ______________________________

From Scoring Guide for Activity 1 p. 29

Summary	Level
Problem Solving	
Concepts	

From Scoring Guide for Activity 2 p. 37

Summary	Level
Application	

From Scoring Guide for Activity 3 p. 45

Summary	Level
Problem Solving	

Combining the Scores from all Scoring Guides

Summary	Level
Problem Solving	
Concepts	
Application	

Record of Student Achievement on the Grade 8 Unit

What is Shannon's Secret?

Student Name ____

From Scoring Guide for Activity 2 p. 73

Summary	Level
Application	

From Scoring Guide for Activity 4 p. 88

Summary	Level
Communication	

Combining the Scores from all Scoring Guides

Summary	Level
Application	
Communication	

Student Name ____

From Scoring Guide for Activity 2 p. 73

Summary	Level
Application	

From Scoring Guide for Activity 4 p. 88

Summary	Level
Communication	

Combining the Scores from all Scoring Guides

Summary	Level
Application	
Communication	

Student Name ____

From Scoring Guide for Activity 2 p. 73

Summary	Level
Application	

From Scoring Guide for Activity 4 p. 88

Summary	Level
Communication	

Combining the Scores from all Scoring Guides

Summary	Level
Application	
Communication	

Student Name ____

From Scoring Guide for Activity 2 p. 73

Summary	Level
Application	

From Scoring Guide for Activity 4 p. 88

Summary	Level
Communication	

Combining the Scores from all Scoring Guides

Summary	Level
Application	
Communication	

Student Name ____

From Scoring Guide for Activity 2 p. 73

Summary	Level
Application	

From Scoring Guide for Activity 4 p. 88

Summary	Level
Communication	

Combining the Scores from all Scoring Guides

Summary	Level
Application	
Communication	

Additional Resources for Geometry & Spatial Sense

Bartels, Bobbye. "Truss(t)ing Triangles." Focus Issue of *Mathematics Teaching in the Middle School*, Vol 3, #6. Reston, VA: National Council of Teachers of Mathematics, March-April, 1998, pp. 394-396.

Brahier, Daniel J., and William R. Speer. "Worthwhile Tasks: Exploring Mathematical Connections Through Geometric Solids." *Mathematics Teaching in the Middle School*, Vol 3, #1. Reston, VA: National Council of Teachers of Mathematics, September, 1997, pp. 20-28.

Geddes, Dorothy et al. *Geometry in the Middle Grades.* From Addenda Series, Grades 5–8: Curriculum and Evaluation Standards for School Mathematics, Reston, VA: National Council of Teachers of Mathematics, 1992.

Harrell, Marvin E., and Linda S. Fosnaugh. "Allium to Zircon: Mathematics & Nature." *Mathematics Teaching in the Middle School*, Vol 2, #6. Reston, VA: National Council of Teachers of Mathematics, May, 1997, pp. 380-389.

Manouchehri, Azita et al. "Exploring Geometry with Technology." Focus Issue of *Mathematics Teaching in the Middle School*, Vol 3, #6. Reston, VA: National Council of Teachers of Mathematics, March-April, 1998, pp. 436-442.

Sundberg, Sue. "A Plethora of Polyhedra." Focus Issue of *Mathematics Teaching in the Middle School*, Vol 3, #6. Reston, VA: National Council of Teachers of Mathematics, March-April, 1998, pp. 388-391.

Zilliox, Joseph T., and Shannon G. Lowrey. "Many Faces Have I." *Mathematics Teaching in the Middle School*, Vol 3, #3. Reston, VA: National Council of Teachers of Mathematics, Nov.-Dec., 1997, pp. 180-183.

Videotapes

Apostol, Tom. *The Theorem of Pythagoras: Project Mathematics!* California Institute of Technology, 1988.

———— *The Story of π: Project Mathematics!* California Institute of Technology, 1989.

Web Sites

A Teacher's Guide to Building the Icosahedron	http://www.geom.umn.edu/docs/education/build-icos/
Interactive Geometry Classroom Resources	http://forum.swarthmore.edu/dynamic.html
	http://forum.swarthmore.edu/sketchpad/sketchpad.html
Math Forum – Elementary Problem of the Week	http://forum.swarthmore.edu/elempow/
Math Forum – Geometry Problem of the Week	http://forum.swarthmore.edu/geopow/
Math Forum: Project of the Month	http://forum.swarthmore.edu/pom2/index.html
Eisenhower National Clearinghouse	http://www.enc.org/

Free Software for Ontario Schools

The Ministry of Education and Training of Ontario purchases site licences of software for all publically funded schools in the province. This software can be obtained from the Ontario Educational Software Service (OESS) representative in your school district. To determine what is available, access this web site: **http://www.tvo.org/osapac**